Lecture 3: Solution Concept and Well-posedness of Hybrid Systems

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What we will see together

Lecture 3: Solution Concept and Well-posedness of Hybrid Systems

Lecture 4: Stability and Robustness of Hybrid Systems

Lecture 5: Hybrid Systems and Control

May, 9 and May, 16

1 assignment

Modeling framework: hybrid inclusion

In the next two (even three) lectures, we concentrate on hybrid systems modeled as in [Goebel et al., 2012], i.e.

$$\begin{cases} \frac{d}{dt}x = \dot{x} \in F(x) & x \in C \\ x(t^+) = x^+ \in G(x) & x \in D, \end{cases} \qquad (\mathcal{H})$$

where

- $C \subseteq \mathbb{R}^n$ is the **flow set**
- $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the **flow map**
- $D \subseteq \mathbb{R}^n$ is the **jump set**
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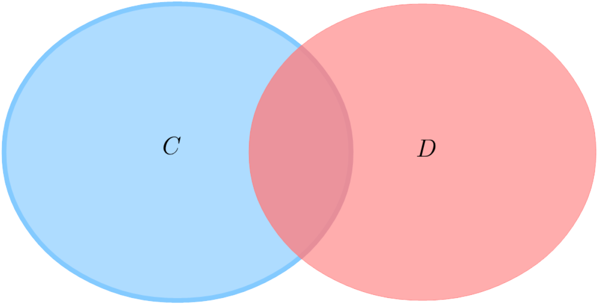
Modeling framework: main idea

$$\dot{x} \in F(x) \quad x \in C \qquad\qquad x^+ \in G(x) \quad x \in D. \tag{$\mathcal{H}$}$$



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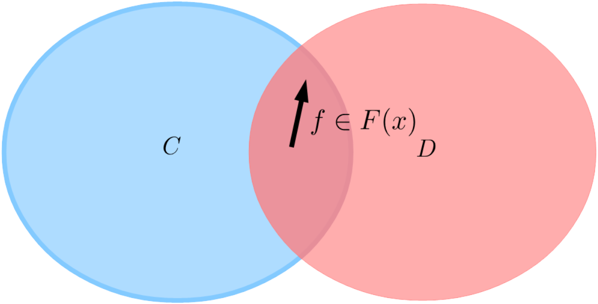

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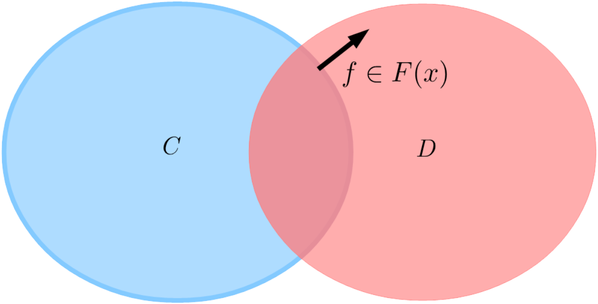
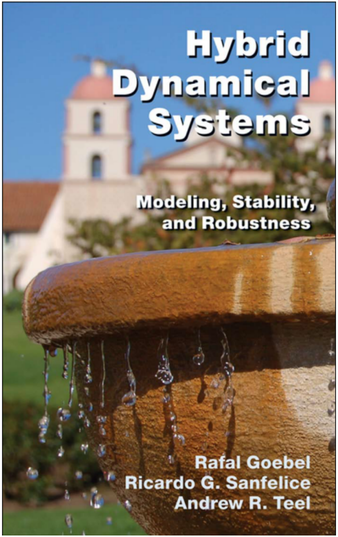


Modeling framework: the book

R. Goebel, R. Sanfelice and A. Teel, *Hybrid Dynamical Systems: Modeling, Stability and Robustness*, Princeton University Press, 2012.



Modeling framework: remarks

$$\begin{cases} \dot{x} & \in & F(x) & x \in C \\ x^+ & \in & G(x) & x \in D, \end{cases} \qquad (\mathcal{H})$$

- Why "$\in$" and not "$=$"?
- Why no external inputs?

  If the input is a feedback, i.e., $u = K(x)$, we obtain $\mathcal{H}$.
  If the input is exogenous, things are a bit more involved... but there are results in the literature
- Why no time-dependency, i.e., $\dot{x} \in F(t, x)$ or $x^+ \in G(k, x)$?

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  We can always extend the state as $z = (x, t)$ so that

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix} \in \begin{pmatrix} F(t,x) \\ 1 \end{pmatrix} = \widetilde{F}(z)$$
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  We can always extend the state as $z = (x,k)$ so that

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$$z^+ = \begin{pmatrix} x^+ \\ k^+ \end{pmatrix} \in \begin{pmatrix} G(k,x) \\ k+1 \end{pmatrix} = \widetilde{G}(z)$$

Modeling framework: questions

Recall

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \qquad (\mathcal{H})$$

Questions

- What do we mean by a solution?
- Can new phenomena arise with $\mathcal{H}$?
- When do we know whether a solution exists?
- Can we say something about uniqueness of solutions? Is it really important?
- What do we mean by a well-posed hybrid systems?

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Overview

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Differential equations: the linear case

Let us go back to the basics and consider the linear time-invariant system

$$\dot{x} = Ax, \tag{LIN}$$

where $A \in \mathbb{R}^{n\times n}$.

Given any initial condition x_0, the corresponding solution to (LIN) is given by, for any $t \geq 0$,

$$x(t) = e^{At}x_0,$$

where $e^{At} = \mathbb{I} + At + A^2\frac{t^2}{2!} + \dots$ is an exponential matrix.

We naturally want to call x a solution as $\dot{x}(t) = Ax(t)$ for all $t \geq 0$.

Remark: for any initial condition, the solution is **unique** and **defined for all times**.

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Differential equations: the nonlinear case

Consider

$$\dot{x} = f(x), \tag{NL}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$.

What is a solution to (NL)?

We may be tempted to call a solution any function $x : [0, \infty) \to \mathbb{R}^n$ such that, for any $t \in [0, \infty)$,

$$x(t) = x(0) + \int_0^t f(x(\tau))d\tau$$

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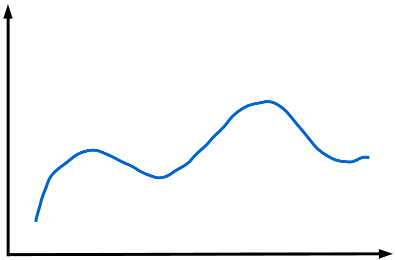

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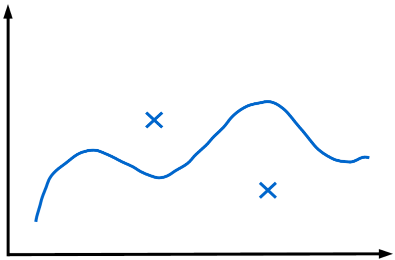

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Definition

Given a time-interval $[0, T)$ with $T \in (0, \infty]$ and an initial value x_0, we say that $x : [0, T) \to \mathbb{R}^n$ is a solution to (NL) on $[0, T)$ initialized at $x_0 \in \mathbb{R}^n$ if it is **continuous**, **differentiable** and satisfies:

- $x(0) = x_0$
- $\dot{x}(t) = f(x(t))$ for all $t \in [0, T)$

Sufficient conditions for the existence of solutions: f continuous.

Remark: We consider solution on $[0, T)$, the definition can be straightforwardly adapted to $[a, b)$ with $-\infty < a \leq b \leq \infty$.

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Differential equations: existence of solutions

The equation $\dot{x} = f(x)$ does not always have a solution defined at any $t \geq 0$.

Consider

$$\dot{x} = x^2, \quad x_0 > 0$$

We can solve this equation and obtain

$$x(t) \quad = \quad \frac{x_0}{1 - t x_0}.$$

We note that $x(t) \to \infty$ as $t \to \dfrac{1}{x_0}$: **explosion in finite-time.**

The solution is only defined on $[0, \dfrac{1}{x_0})$.

Sufficient conditions to avoid this[1]:

- f globally Lipschitz, i.e., there exists $L \geq 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $(x, y) \in \mathbb{R}^n$.
- OR if a solution is guaranteed to remain in a bounded set on its domain of existence, say $[0, T)$, then this solution is defined (actually, can be extended) for all $t \geq 0$, i.e., $T = \infty$

[1] See [Khalil, 2002]

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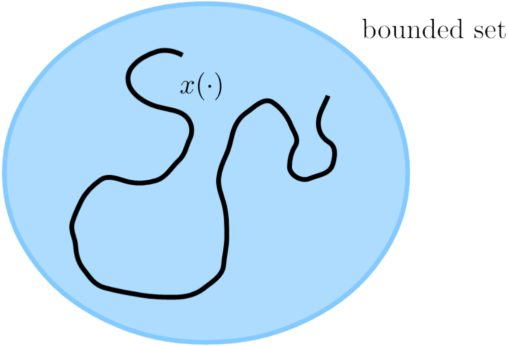
Differential equations: the nonlinear case



Differential equations: non-unique solutions

Solutions, when they exist, are not always unique

Consider

$$\dot{x} = x^{1/3} \text{ with } x(0) = 0$$

There are two solutions $x(t) = 0$ and $x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$.

Sufficient conditions for uniqueness[2]**:**

- f is locally Lipschitz, i.e., for any bounded set $B \subseteq \mathbb{R}^n$, there exists $L(B) \geq 0$ such that $|f(x) - f(y)| \leq L(B)|x - y|$ for all $(x, y) \in B^2$
- f is continuously differentiable, which implies f is locally Lipschitz (by application of the mean value theorem).

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Differential equations

We are going to face these phenomena when dealing with hybrid systems

Differential inclusions: a piecewise linear example

Consider

$$\dot{x} = f(x), \qquad \text{(NL-disc)}$$

What happens when f is discontinuous?

Differential inclusions: a piecewise linear example

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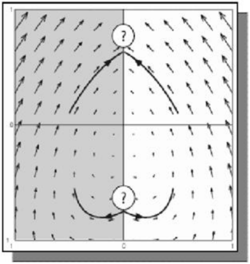
$$\dot{x} = f(x), \qquad \text{(NL-disc)}$$

What happens when f is discontinuous?

Consider the piecewise linear system

$$\dot{x} = \begin{cases} A_1 x, & \text{when } x_1 < 0 \\ A_2 x, & \text{when } x_1 > 0, \end{cases}$$

where $A_1 = \begin{pmatrix} -2 & 2 \\ -4 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -2 & -2 \\ 4 & 1 \end{pmatrix}$



(Courtesy from Maurice Heemels)

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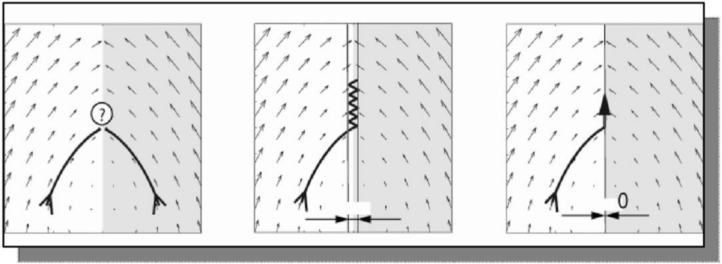

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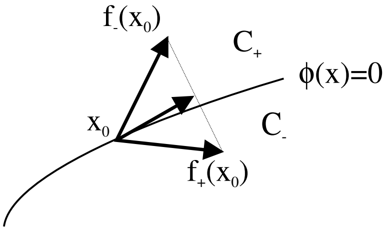
Differential inclusions: sliding mode

More generally



$f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+.

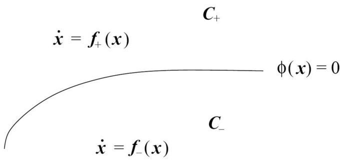
How should we define the solution?

Filippov convexication: when $\phi(x) = 0$,

$$\dot{x} = \lambda f_+(x) + (1 - \lambda) f_-(x) \text{ with } \lambda \in [0, 1)$$

$\rightarrow$ "third mode arises"

Differential inclusions: sliding mode



$$\dot{x} = \begin{cases} f_+(x), & \text{when } \phi(x) > 0 \\ f_-(x), & \text{when } \phi(x) < 0. \end{cases}$$

- x in the interior of C_- or C_+: just flow!
- $f_-(x)$ and $f_+(x)$ point in the same direction: just flow!
- $f_+(x)$ points towards C_+ and $f_-(x)$ points towards C_-: at least two solutions.
- $f_+(x)$ points towards C_- and $f_-(x)$ points towards C_+: sliding mode $\rightarrow$ convexification.

Differential inclusions: sliding mode

As a result

$$\dot{x} = \begin{cases} f_+(x), & \text{when } \phi(x) > 0 \\ f_-(x), & \text{when } \phi(x) < 0 \\ \lambda f_+(x) + (1-\lambda) f_-(x), & \text{when } \phi(x) = 0, \text{ for some } \lambda \in [0,1) \end{cases} \qquad \text{(DISC)}$$

We do have a set when $\phi(x) = 0 \Rightarrow \dot{x} \in F(x)$.

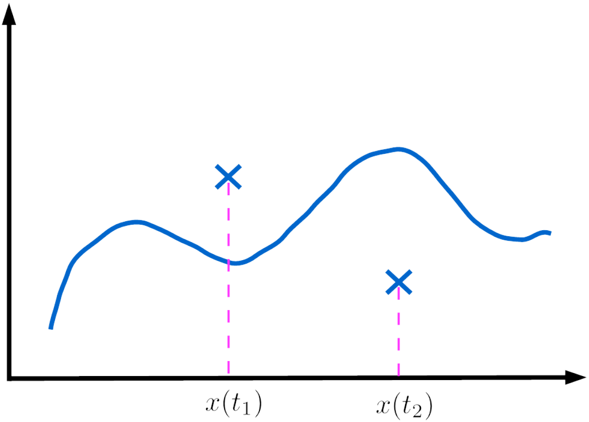
Definition

Given a time-interval $[0, T)$ with $T \in (0, \infty]$ and an initial value x_0, we say that $x : [0, T) \to \mathbb{R}^n$ is a solution to (DISC) on $[0, T)$ initialized at $x_0 \in \mathbb{R}^n$ if it is **absolutely continuous** on $(0, T)$ and satisfies:

- $x(0) = x_0$
- $\dot{x}(t) \in F(x(t))$ for almost all $t \in [0, T]$

Differential inclusions: “almost everywhere”

“almost all t = for all t except those in a set of Lebesgue measure zero”



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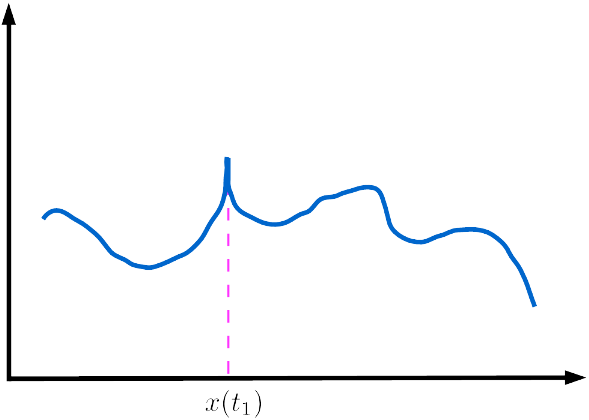

Differential inclusions: absolute continuity

We say that $f : [0, T] \to \mathbb{R}^n$ is **absolutely continuous** if:

- f has a derivative f' almost everywhere on $[0, T]$
- $f(t) = f(0) + \int_0^T f'(\tau) d\tau$ for all $t \in [0, T]$.
  "*An absolutely continuous function coincides with the integral of its derivative.*"

Absolute continuity $\Rightarrow$ continuity

Difference equations: easy!

Consider

$$x^+ \in G(x) \qquad \text{(DT-incl)}$$

where $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

Definition

Given an initial value x_0, we say that $x : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ is a solution to (DT-incl) initialized at $x_0 \in \mathbb{R}^n$ at time 0 if

- $x(0) = x_0$,
- $x(j+1) \in G(x(j))$ for all $j \in \mathbb{Z}_{\geq 0}$.

Good properties:

- Solution are defined for all positive time $i \in \mathbb{Z}_{\geq 0}$ (it cannot explode in finite-time).
- Solutions are unique if $G(x)$ is single-valued for any $x \in \mathbb{R}^n$.

Difference equations: examples

- $x^+ = x^2$ for $x \in \mathbb{R}$

  $x(1) = x_0^2$

  $x(2) = x(1)^2 = x_0^4$ etc.

- (Jump counter) $x^+ = x + 1$ for $x \in \mathbb{Z}_{\geq 0}$

  $x(1) = x(0) + 1$

  $x(2) = x(1) + 1 = x(0) + 2$

  $x(k) = x(0) + k$ for $k \in \mathbb{Z}_{\geq 0}$.

- (Set-valued case) $x^+ \in [T_{\min}, T_{\max}]$ where $-\infty < T_{\min} \leq T_{\max} < \infty$

  $x(1) \in [T_{\min}, T_{\max}]$
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Overview

New phenomena: bye bye C and D

It is time to move to hybrid systems.

Consider

$$\begin{cases} \dot{x} & = & 1 & \qquad C = [0,1] \\ x^+ & = & 2 & \qquad D = \{1\}. \end{cases}$$

Let $x = 0$.

- At $t = 1$, $x(t) = 1$.
- A jump occurs at $t = 1$ and x becomes equal to 2.
- x has left C and D $\to$ it stops to exist.

Solution may not be defined for all times because it may leave $C \cup D$, in which case we can no longer define it.

How to avoid this?

When $G(D) \subseteq C \cup D$, after a jump, we remain in $C \cup D$.

Here $G(D) = \{2\} \nsubseteq C \cup D = [0,1]$.

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New phenomena: stuck at the border

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- A jump cannot occur at $t = 1$ as $D = \{2\}$ and it cannot flow as, if we would keep flowing, x would leave set C

$\rightarrow$ x **is stuck at the border of** C**!**

How to avoid this?

By requiring that, the solution can flow when $x \in C \backslash D$. We will formalize this later using the concept of tangent cone.

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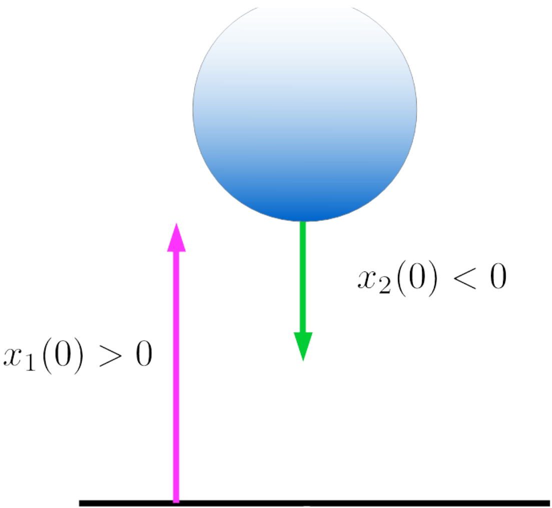

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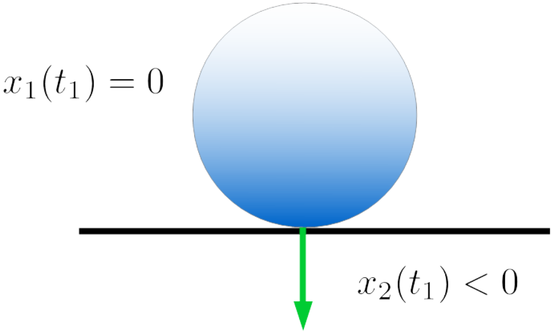

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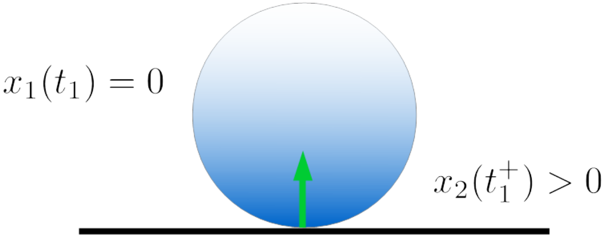

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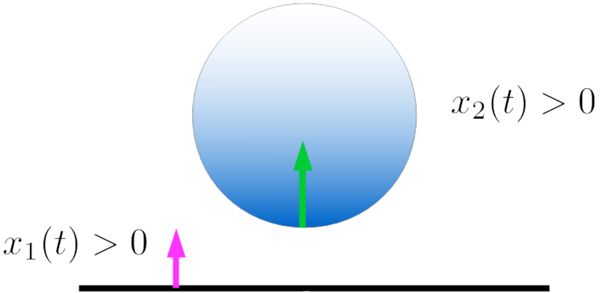


New phenomena: Zeno phenomenon

Bouncing ball where a jump corresponds to an impact of the ball

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x_1 > 0 \text{ or } (x_1 = 0 \text{ and } x_2 > 0) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} \qquad x_1 = 0 \text{ and } x_2 < 0$$

where

- $x_1 \in \mathbb{R}$ is the height
- $x_2 \in \mathbb{R}$ is the vertical velocity
- $\gamma > 0$ is the acceleration due to gravity
- $\lambda \in (0, 1)$

New phenomena: Zeno phenomenon

If $x_1(0) = 0$ and $x_2(0) > 0$, the jumps times t_i, $i \in \mathcal{J} \subseteq \mathbb{Z}_{\geq 0}$, are related through

$$t_{i+1} = t_i + \frac{2\lambda^i}{\gamma} x_2(0)$$

Hence, assuming $t_0 = 0$,

$$\begin{aligned} t_i &= \sum_{j=0}^{i} \frac{2\lambda^i}{\gamma} x_2(0) \\ &= \frac{2}{\gamma} x_2(0) \frac{1-\lambda^{i+1}}{1-\lambda} \end{aligned}$$

Consequently,

$$t_i \to \frac{2}{\gamma} x_2(0) \frac{1}{1-\lambda} \text{ as } i \to \infty$$

The solution jumps infinitely many times in finite time: Zeno phenomenon

We will see that this is not an issue with the hybrid formalism in terms of existence of solutions.

Overview

The solution concept: key idea

System

$$\begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \qquad \begin{array}{l} \text{continuous time } t \in \mathbb{R}_{\geq 0} \\ \text{discrete time } j \in \mathbb{Z}_{\geq 0} \end{array} \qquad (\mathcal{H})$$

Key idea

To parameterize solutions both by t and j.

We therefore write

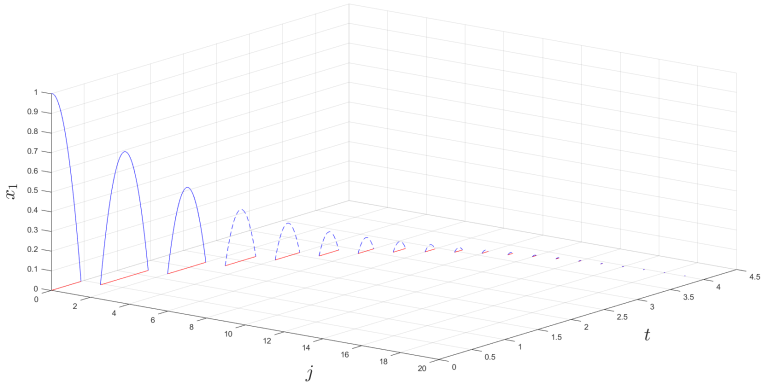
$$x(t, j)$$

$\rightarrow$ we keep track of

- the elapsed continuous time
- and the number of jumps the solution has experienced.

The solution concept: key idea

Bouncing ball (x_1 component)



The solution concept: key idea

We will no longer write $x(t)$ for $t \in [0, T)$ where $T \in (0, \infty]$

We write instead $x(t, j)$ for

$$(t, j) \in \bigcup_{i=0}^{J} [t_i, t_{i+1}] \times \{i\},$$

or equivalently

$$(t, j) \in \bigcup_{i=0}^{J} ([t_i, t_{i+1}], i)$$

The solution concept: the domain of a set-valued map

Definition

Given a set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the **domain** of M is the set

$$\text{dom}\, M = \{x \in \mathbb{R}^m \;:\; M(x) \neq \emptyset\}.$$

Examples:

- $x^+ = x^2 = G(x)$, dom $G = \mathbb{R}$, $x(j) = x(0)^{2j}$, dom $x = \mathbb{Z}_{\geq 0}$.
- $\dot{x} = x^2 = F(x)$, dom $F = \mathbb{R}$, $x(t) = \frac{x_0}{1 - tx_0}$, dom $x = [0, \frac{1}{x_0})$ when $x_0 > 0$.
- $F(x) = S \neq \emptyset$ when $x \in C$ and $F(x) = \emptyset$ when $x \in \mathbb{R}^n \backslash C$, dom $F = C$.

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The solution concept: hybrid time domains

Definition

A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a **compact hybrid time domain** if

$$E = \bigcup_{j=0}^{J-1} \left([t_j, t_{j+1}], j \right)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_J$. It is a **hybrid time domain** if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

A solution ϕ to a hybrid system is defined on/associated to a hybrid time domain denoted $\operatorname{dom} \phi$

The solution concept: hybrid arc

Definition

A function $\phi : E \to \mathbb{R}^n$ is a **hybrid arc** if:

- E is a hybrid time domain,
- for each $j \in \mathbb{Z}_{\geq 0}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j = \{t : (t, j) \in E\}$.

- I^j? If we denote the continuous time instants at which a jump occurs as t_j, $I^j = [t_j, t_{j+1}]$.
- locally absolutely continuous? absolutely continuous on each compact subinterval of I^j.

The solution concept: types of hybrid arcs

Some notations

$$\sup_t \text{dom}\,\phi := \sup\left\{t \in \mathbb{R}_{\geq 0} \;:\; \exists j \in \mathbb{Z}_{\geq 0} \text{ such that } (t,j) \in \text{dom}\,\phi\right\}$$

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Definition

A hybrid arc ϕ is called

- **nontrivial** if dom ϕ contains at least two points.
- **complete**, if dom ϕ is unbounded, i.e.

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- t-**complete** if $\sup_t \text{dom}\,\phi = \infty$.
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The solution concept: a detour to the basic assumptions (simplified)

Consider the hybrid differential **equation**

$$\dot{x} = f(x) \quad x \in C, \qquad x^+ = g(x) \quad x \in D.$$

Basic assumptions

Flow and jump sets:

- C and D are closed subsets of $\mathbb{R}^n$.

Flow map $f : \mathbb{R}^n \to \mathbb{R}^n$:

- f is continuous,
- $C \subset \operatorname{dom} f$.

Jump map $g : \mathbb{R}^n \to \mathbb{R}^n$:

- g continuous,
- $D \subset \operatorname{dom} g$.

The solution concept: a detour to the basic assumptions

We are going to generalize these assumptions to differential and difference **inclusions**, i.e., $\dot{x} \in F(x)$ and $x^+ \in G(x)$

When F is single-valued (we denote it f), f has to be continuous. We ask for similar properties when F is set-valued.

Continuity of f means that for any converging sequence $x_n \to x$ as $n \to \infty$, $f(x_n) \to f(x)$ as $n \to \infty$.

$\to$ We ask for a similar property for F, namely that for any $x_n \to x$ as $n \to \infty$, and any converging sequence $u_n \in F(x_n)$, with $u_n \to u$ as $n \to \infty$, $u \in F(x)$: we talk of **outer-semicontinuity**.

When $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, we also know that for any x, there exists a bounded neighborhood of x denoted $\mathcal{B}$, such that $f(\mathcal{B})$ is bounded.

$\to$ We ask for a similar property for F: we talk of **local boundedness**

We need, in addition, some convexity properties for F to be able to define solutions as we saw to cope with sliding modes.

Similar conditions apply to G, except the convexity property.

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We need, in addition, some convexity properties for F to be able to define solutions as we saw to cope with sliding modes.

Similar conditions apply to G, except the convexity property.

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The solution concept: a detour to the basic assumptions

Recall

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D.$$

Basic assumptions

Flow and jump sets:

- C and D are closed subsets of $\mathbb{R}^n$.

Flow map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:

- F is outer semi-continuous,
- F is locally bounded relative to C,
- $C \subset \operatorname{dom} F$,
- $F(x)$ is convex for every $x \in C$.

Jump map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$:

- G is outer semi-continuous,
- G is locally bounded relative to D,
- $D \subset \operatorname{dom} G$.

The solution concept: outer semicontinuity

A set-valued map $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is **outer semicontinuous** at $x \in \mathbb{R}^m$ if for any sequence x_n, $n \in \mathbb{Z}_{\geq 0}$ converging to x, and any sequence $u_n \in F(x_n)$ converging to u, then $u \in F(x)$.

Equivalent characterization: F is outer semicontinuous if $\{(x, z) \, : \, z \in F(x)\}$ is closed.

The solution concept: outer semicontinuity

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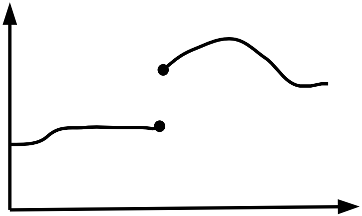


The solution concept: local boundedness

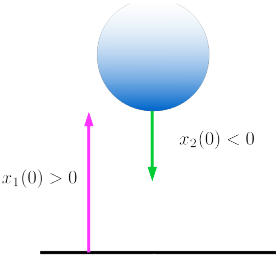
A set-valued map $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is **locally bounded** at x, if there exists a neighborhood U of x such that $F(U)$ is bounded.

The solution concept: a detour to the basic assumptions

Bouncing ball

$$\dot{x} = \begin{cases} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x_1 > 0 \text{ or } (x_1 = 0 \text{ and } x_2 > 0) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

$$x^+ = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} \quad x_1 = 0 \text{ and } x_2 < 0$$



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The flow and jump sets are not closed because of strict inequalities, so we redefine these are

$$C = \{(x_1, x_2) \, : \, x_1 \geq 0\}$$

$$D = \{(x_1, x_2) \, : \, x_1 = 0 \text{ and } x_2 \leq 0\}$$

The jump map is continuous ✓

The flow map is not outer-semicontinuous at $x = 0$. Indeed, $f(x) \to (0, -\gamma)$ as $x \to 0$, while $f(0) = (0, 0)$. We can regularize it as follows

$$F(x) = \begin{cases} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x \neq 0 \\ \begin{pmatrix} 0 \\ \{-\gamma, 0\} \end{pmatrix} & x = 0 \end{cases}$$

Now F is convex, outer-semicontinuous and locally bounded as desired.

The solution concept: a detour to the basic assumptions

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The solution concept: hybrid solution

Recall

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D. \qquad (\mathcal{H})$$

Definition

A hybrid arc ϕ is a solution to $\mathcal{H}$ if:

- $\phi(0,0) \in C \cup D$,
- for every $j \in \mathbb{Z}_{\geq 0}$, for almost all $t \in I^j = \{t : (t,j) \in \operatorname{dom}\phi\}$ ("$[t_j, t_{j+1}]$"),

$$\phi(t,j) \in C \text{ and } \dot{\phi}(t,j) \in F(\phi(t,j)),$$

- for every $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,

$$\phi(t,j) \in D \text{ and } \phi(t,j+1) \in G(\phi(t,j)).$$

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The solution concept: conditions for the existence of non-trivial solutions

Recall: non-trivial solution means $\operatorname{dom}\phi$ contains at least two points

When are we sure solutions are non-trivial?

Proposition

Let $\xi \in C \cup D$. If

- $\xi \in D$
- OR $\xi \in C \backslash D$ and there exists a neighborhood U of ξ such that for any $x \in U \cap C$,

$$F(x) \cap T_C(x) \neq \emptyset, \qquad \text{Viability Condition (VC)}$$

then there exists a nontrivial solution ϕ with $\phi(0,0) = \xi$.

Tangent cone: The tangent cone to C at a $x \in \mathbb{R}^n$, denoted $T_C(x)$, is the set of all vectors $w \in \mathbb{R}^n$ for which there exist $x_i \in C$, $\tau_i > 0$ with $x_i \to x$ and $\tau_i \searrow 0$ as $i \to \infty$ and

$$w = \lim_{i \to \infty} \frac{x_i - x}{\tau_i}.$$

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The solution concept: viability condition

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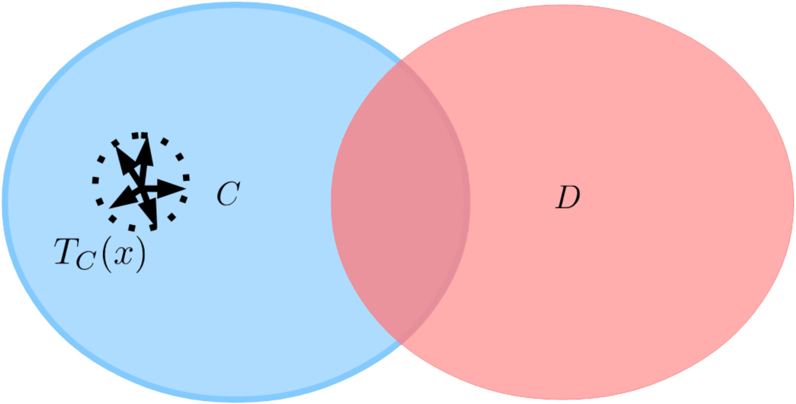

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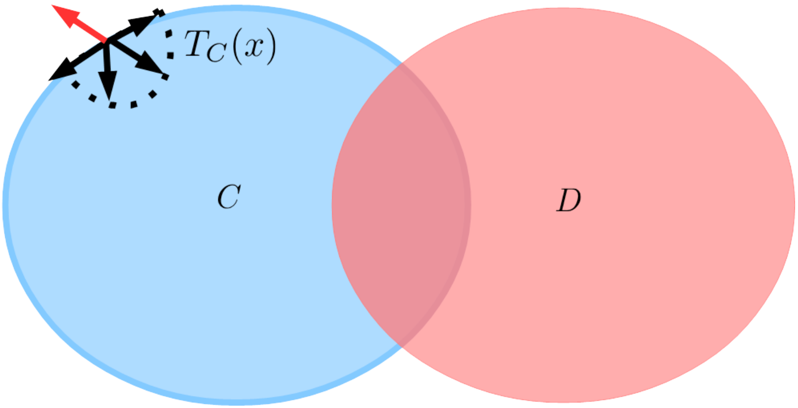
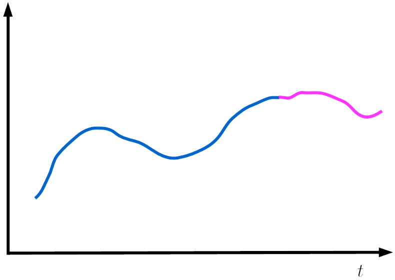


The solution concept: maximal solutions

Can we say more about non-trivial solutions?

Definition

A solution ϕ to hybrid system is **maximal** if there exist no other solutions ψ such that $\operatorname{dom}\phi$ is a (proper) subset of $\operatorname{dom}\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$.



The solution concept: about non-trivial solutions

Proposition

If (VC) holds for any $\xi \in C \backslash D$, then there exists a nontrivial solution from every point in $C \cup D$, and every maximal solution satisfies one of the next conditions:

- (complete) ϕ is complete;
- (explosion in finite-time) ϕ explodes in finite continuous-time, in particular $\operatorname{dom} \phi$ is bounded and I^J where $J = \sup_j \operatorname{dom} \phi$ has nonempty interior and $t \mapsto |\phi(t, J)| \to \infty$ as $t \to \sup_t \operatorname{dom} \phi$;
- (jumps outside $C \cup D$) $\phi(T, J) \notin C \cup D$ where $(T, J) = \sup \operatorname{dom} \phi$.

The solution concept: examples

Consider again system

$$\begin{cases} \dot{x} & = & 1 & \quad C = [0,1] \\ x^+ & = & 2 & \quad D = \{1\}. \end{cases}$$

Viability condition: let $x \in C \backslash D = [0,1)$,

- $T_C(0) = [0,\infty)$, consequently $T_C(0) \cap F(0) = [0,\infty) \cap \{1\} \neq \emptyset$ ✓
- $T_C(x) = \mathbb{R}$ for any $x \in (0,1)$, consequently $T_C(x) \cap F(x) = [0,\infty) \cap \{1\} \neq \emptyset$ ✓

VC holds!

We have already seen that $G(D) \not\subseteq C \cup D$: one of the conditions of the proposition holds.

The solution concept: examples

Consider again system

$$\left\{ \begin{array}{lllr} \dot{x} & = & 1 & C = [0,1] \\ x^+ & = & 2 & D = \{2\}. \end{array} \right.$$

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- $T_C(x) = \mathbb{R}$ for any $x \in (0,1)$, consequently $T_C(x) \cap F(x) = [0,\infty) \cap \{1\} \neq \emptyset$ ✓
- $T_C(1) = (-\infty,0]$, consequently $T_C(x) \cap F(1) = (-\infty,0] \cap \{1\} = \emptyset$ ✗

VC does not hold!

The system has a trivial solution at $x = 1$.

The solution concept: examples

Regularized bouncing ball

$$\left\{\begin{array}{lcll} \dot{x} & \in & \left\{\begin{array}{ll} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x \neq 0 \\ \begin{pmatrix} 0 \\ [-\gamma, 0] \end{pmatrix} & x = 0 \end{array}\right. & x_1 \geq 0 \\ x^+ & = & \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} & x_1 = 0 \text{ and } x_2 \leq 0. \end{array}\right.$$

Viability condition: let $x \in C \backslash D$,

- when $x_1 > 0$, $T_C(x) = \mathbb{R}^2$, consequently $T_C(x) \cap F(x) = F(x) \neq \emptyset$ ✓
- when $x_1 = 0$ and $x_2 > 0$, $T_C(x) = [0, \infty) \times \mathbb{R}$, consequently $T_C(x) \cap F(x) = ([0, \infty) \times \mathbb{R}) \cap (0, -\gamma) = (0, -\gamma) \neq \emptyset$ ✓

VC holds!

$G(D) = \{0\} \times ... \subset C \cup D$.

Solutions cannot explode in finite (continuous) time because the flow dynamics are affine.

We conclude that maximal solutions are complete, in particular they are Zeno as we know, which is not an issue here.

The solution concept: uniqueness of solutions

Can we say something about uniqueness of solutions?

Such conditions exist.

Not always a good idea when dealing with hybrid inclusions.

Often difficult to ensure the basic conditions (and thus robustness, see Lecture 4) without sacrificing uniqueness of solutions.

Overview

Well-posed hybrid systems: a very short introduction

Traditionally, we say that a dynamical system is **well-posed** when

- it generates a solution for any initial condition ✓,
- this solution is unique,
- solution continuously depend on parameters (and thus on initial conditions).

Too much to ask for hybrid systems and not necessary to build up a solid stability theory.

Well-posed hybrid systems: a very short introduction

Notation: For any $x \in C \cup D$, $\mathcal{S}(x)$ is the set of all solutions initialized at x

Essentially, we say that a hybrid system is:

- **nominally well-posed** if, when we take a (graphically) converging sequence of initial conditions x_n, the corresponding sequence of solution $\phi_n \in \mathcal{S}(x_n)$ (which may be a set because of non-unique solutions) (graphically) converge to a solution.
- **well-posed** if this property remains true even if the system is perturbed by vanishing disturbances.

Fundamental property for robustness and in a number of proofs.

We do not have to worry about it thanks to the basic conditions.

Theorem

If a hybrid system satisfies the basic conditions, it is (nominally) well-posed.

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If a hybrid system satisfies the basic conditions, it is (nominally) well-posed.

Well-posed hybrid systems: ρ-perturbed hybrid system

ρ-perturbation of the hybrid system

$\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (continuous typically)

$$\begin{cases} \dot{x} & \in & F_\rho(x) & & x \in C_\rho \\ x^+ & \in & G_\rho(x) & & x \in D_\rho, \end{cases} \qquad (\mathcal{H}_\rho)$$

where

$$\begin{array}{lcl} C_\rho & = & \{x \,:\, (x+\rho(x)\mathbb{B}) \cap C \neq \emptyset\} \text{ “} = C \text{ inflated by something of the order of } \rho(x)\text{''} \\ D_\rho & = & \{x \,:\, (x+\rho(x)\mathbb{B}) \cap D \neq \emptyset\} \text{ “} = D \text{ inflated by something of the order of } \rho(x)\text{''} \end{array}$$

$$F_\rho(x) = \overline{\text{con}} F\left((x+\rho(x)\mathbb{B}) \cap C\right) + \rho(x)\mathbb{B} \ \forall x \in \mathbb{R}^n, \text{ “} = f(x+\rho(x)) + \rho(x)\text{''}$$

$$\begin{array}{lcl} G_\rho(x) & = & \{v \in \mathbb{R}^n \,:\, v \in g + \rho(g)\mathbb{B},\, g \in G\left((x+\rho(x)\mathbb{B}) \cap D\right)\} \ \ \forall x \in \mathbb{R}^n \\ & = & \text{“}g(x+\rho(x)) + \rho(x)\text{''}. \end{array}$$

and

- $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$
- later, when talking of stability $\rho(x) > 0$ when x not in the attractor

Overview

Summary

- Recall on solutions for differential/difference equations/inclusions
- Special care is needed when studying hybrid inclusions
- Notion of solutions for hybrid inclusions
- Conditions for existence of solutions
- Few words on the notion of well-posedness

We are ready to talk of stability

Summary: references

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