Asynchronous mix-triggered control

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***Abstract*— Asynchronous event-triggered control (AETC) is a triggering strategy for the feedback channel of a closed-loop control system. AETC aims at reducing transmissions compared with time-triggered control strategies and listening time compared with other event-triggered control strategies. This work is an extension of the asynchronous event-triggered control [6] on reducing periodic listening time spent for the threshold update signal. In this work, by introducing an autonomous time-varying threshold update mechanism at every node, the listening time can totally be removed while still guaranteeing a pre-designed control performance. A numerical example is shown to illustrate the developed strategy.**

I. INTRODUCTION

In digital control implementations, the control task includes: plant output sampling and transmition, and controller output computation and application. All these processes consume system resources. These resources may include wireless channel bandwidth, mobile node energy, or CPU utilization rate, among others. In networked control systems (NCSs), especially wireless networked control systems (WNCSs), it is important to save these system resources to attain reduced costs of deployment and operation.

Compared with the traditional time-triggered control (TTC) task execution model, in which the tasks are executed periodically, event-triggered control (ETC) task execution, such as [1] [2] [4] [5] [6] [8] [10] [11] [12], can reduce transmissions in the feedback channel and controller output computations.

In [6], an asynchronous event-triggered control (AETC) strategy is presented. This strategy has two event-triggered mechanisms: a sampling update mechanism and a threshold update mechanism. By comparing local measurements of the state and a local threshold, each sensor node can determine local events and update the corresponding sampled-and-hold state in the controller independently of each other. The controller updates the threshold based on the current sampled state and broadcasts this update signal to all sensor nodes. Only one bit is required in each transmission, indicating the existence of the event and the sign of the error (for the sampling update mechanism). Therefore, the required number of transmissions, and bit length of each packet can both be reduced. The work in [3] presents an algorithm to maximize the sampling intervals of the states while guaranteeing a pre-designed performance.

This AETC has been proven to reduce transmissions compared to TTC guaranteeing the same performance [7]; while it does not require the sensor nodes continuously listening to the feedback channel as other ETC strategies. However, it still requires the sensor nodes listening for the threshold update signal periodically, which still consumes a significant amount of energy.

In this work, we replace the event-based threshold update mechanism by an autonomous time-varying one. By autonomous time-varying threshold update mechanism we refer to triggering thresholds governed by an autonomous differential equation. Since the sampling update mechanism is still event-based, we call this triggering mechanism as mix-triggered, indicating the combination of event-triggered and autonomous time-varying.

II. NOTATION AND PRELIMINARIES

We denote the positive real numbers by $\mathbb{R}^+$, the positive real numbers together with 0 by $\mathbb{R}_0^+$, the natural numbers including zero by $\mathbb{N}$. $|\cdot|$ denotes the Euclidean norm in the appropriate vector space, when applied to a matrix $|\cdot|$ denotes the l_2 induced matrix norm. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite, denoted by $P \succ 0$, whenever $x^{\mathrm{T}}Px > 0$ for all $x \neq 0$, $x \in \mathbb{R}^n$. A function α: $\mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathcal{K}(\alpha \in \mathcal{K})$ if: α is a continuous function, $\alpha(0) = 0$ and $s_1 > s_2 \Rightarrow \alpha(s_1) > \alpha(s_2)$. A function α: $\mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathcal{K}_\infty(\alpha \in \mathcal{K}_\infty)$ if: $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$. A function α: $\mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathcal{L}(\alpha \in \mathcal{L})$ if: α is a continuous function, $s_1 \geq s_2 \Rightarrow \alpha(s_1) \leq \alpha(s_2)$ and $\lim_{s\to\infty} \alpha(s) = 0$. A function α: $\mathbb{R}^+ \to \mathbb{R}^+$ belongs to class $\mathcal{KL}(\alpha \in \mathcal{KL})$ if: $\forall$(fixed)t : $\beta(\cdot, t) \in \mathcal{K}$ and $\forall$(fixed)s : $\beta(s, \cdot) \in \mathcal{L}$. For matrix P, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the maximum and minimum eigenvalue of P respectively.

Consider a linear time-invariant system

$$\dot{\xi}(t) = A\xi(t) + Bv(t), \tag{1}$$

where $\xi(t) \in \mathbb{R}^n$ is the state vector and $v(t) \in \mathbb{R}^m$ is

This work is partly funded by China Scholarship Council (CSC).

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the input vector at time t. A controller is given by:

$$v(t) = K\xi(t), \tag{2}$$

in which K is given such that $A + BK$ is Hurwitz. The system is completely observable. Each sensor only can access to one of the states. Furthermore, a sample-and-hold implementation is assumed to apply to the controller:

$$v(t) = K\hat{\xi}(t), \tag{3}$$

where

$$\begin{aligned} \hat{\xi}(t) &:= [\hat{\xi}_1(t), \hat{\xi}_2(t), \cdots, \hat{\xi}_n(t)]^{\mathrm{T}} \\ \hat{\xi}_i(t) &:= \xi_i(t^i_{k_i}), t \in [t^i_{k_i}, t^i_{k_i+1}), \forall i = 1, \cdots, n. \end{aligned} \tag{4}$$

Define

$$\varepsilon_i(t) := \hat{\xi}_i(t) - \xi_i(t), \tag{5}$$

the measurement error of each state. This measurement error is the result of the sample-and-hold mechanism.

Before the problem definition, we present some properties of a closed-loop system that we discuss in this paper.

Definition 1. *(Asymptotic Stability) [9]*
The system (1), (3), (4) is said to be uniformly globally asymptotically stable (UGAS) if there exists $\beta \in \mathcal{KL}$ such that for any $t_0 \geq 0$ the following holds:

$$\forall \xi(t_0) \in \mathbb{R}^n, |\xi(t)| \leq \beta(|\xi(t_0)|, t - t_0), \forall t \geq t_0. \tag{6}$$

Definition 2. *(Exponential Stability) [4] [9]*
The system (1), (3), (4) is said to be globally exponentially stable (GES) if there exists $c, a \in \mathbb{R}^+$ such that for any $t_0 \geq 0$ the following holds:

$$|\xi(t, \xi(0))| \leq c|\xi(0)|e^{-at}, \forall t \geq t_0. \tag{7}$$

We call a a lower bound on the decay rate. The system (1), (3), (4) is said to be uniformly globally exponentially stable (UGES) if there exists $c, a \in \mathbb{R}^+$ such that for any $t_0 \geq 0$ the following holds:

$$\forall \xi(t_0) \in \mathbb{R}^n, |\xi(t, \xi(0))| \leq c|\xi(0)|e^{-at}, \forall t \geq t_0. \tag{8}$$

Definition 3. *(Input-to-State Stability) [9]*
The system (1), (3), (4) is said to be (uniformly globally) input-to-state stable (ISS) with respect to v if there exists $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for any $t_0 \in \mathbb{R}^+_0$ the following holds:

$$\begin{aligned} \forall \xi(t_0) \in \mathbb{R}^n, \|v\|_\infty < \infty \Rightarrow \\ |\xi(t)| \leq \beta(|\xi(t_0)|, t - t_0) + \gamma(\|v\|_\infty), \forall t \geq t_0. \end{aligned} \tag{9}$$

The ISS property of a system can also be established by means of ISS-Lyapunov functions, i.e. a system is ISS if and only if a smooth ISS-Lyapunov function exists [9].

Definition 4. *(ISS Lyapunov function) [9]*
A continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}^+_0$ is said to be an ISS Lyapunov function for the closed-loop system (1) (3) (4) if there exists class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, α_V and α_v such that for all $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ the following is satisfied:

$$\begin{aligned} \underline{\alpha}(|\xi(t)|) \leq V(\xi(t)) \leq \overline{\alpha}(|\xi(t)|) \\ \nabla V \cdot f(\xi, v) \leq -\alpha_V \circ V(\xi(t)) + \alpha_v(|v(t)|). \end{aligned} \tag{10}$$

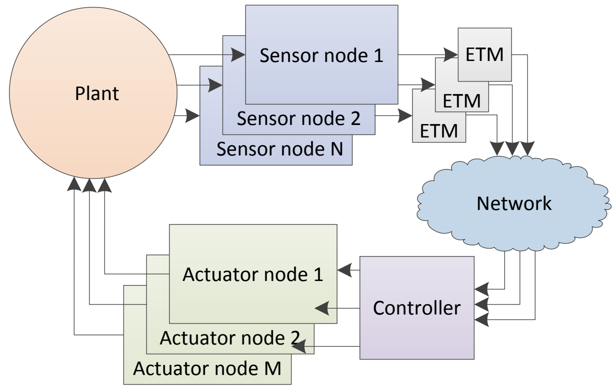

Fig. 1. Networked systems architecture, with decentralized event-triggered mechanism (ETM).

III. Problem definition

The architecture of the system is shown in Figure 1. Assume this system has all of its sensors distributed on a large physical scale. Each sensor is co-located with a network node, formalizing a sensor node. All sensor nodes are connected with the controller via a network. The actuators are co-located with the controller, which is denoted as the controller node.

A decentralized event-triggered control strategy (AETC in our case) is applied to this system. That is, each sensor node is co-located with a local event-triggered mechanism (ETM). Local means, these ETMs are only based on local information, and produce events independently of each other.

The AETC from [6] is reviewed here. There are two event-triggered mechanism in this strategy: sampling update mechanism and threshold update mechanism. The sampling update mechanism executes in each sensor node determines the sampling time of each state:

$$t^i_{k_i} := \min\{t > t^i_{k_i-1} | \varepsilon_i^2(t) = \eta_i(t)\}, \tag{11}$$

where $\eta_i(t) := \theta_i^2 \eta(t)^2$ is the local threshold; $\eta(t)$ is a global threshold. θ is a pre-designed parameter that determines the computation of each local threshold from the global threshold. It satisfies $|\theta| = 1$.

Assumption 1. *An initial state is pre-stored in the controller such that $\varepsilon_i^2(t) \leq \eta_i(t)$, $i = \{1, \cdots, n\}$ hold.*

If there is a sampling event at sensor i, the corresponding estimated state in the controller is then updated as:

$$\hat{\xi}_i(t^i_{k_i}) = \hat{\xi}_i(t^i_{k_i-1}) + \mathrm{sign}(\varepsilon_i(t^i_{k_i}))\sqrt{\eta_i(t^i_{k_i})}. \tag{12}$$

The threshold update mechanism executed in the controller determines the threshold update time of the global threshold:

$$t^c_{r_c+1} := \min\{t \geq t^c_{r_c} + r\tau_c | r \in \mathbb{N}^+, |\overline{\xi}(t)| \leq \rho\eta(t^c_{r_c})\}, \tag{13}$$

where $\rho \in \mathbb{R}^+$ is a pre-designed parameter, satisfying $\rho s \geq \underline{\alpha}^{-1} \circ \epsilon\alpha_v^{-1} \circ \alpha_e(s) + 2s$, for all $s \in (0, \eta(t_0)]$. The estimate $|\overline{\xi}(t)|$ is an upper bound of $|\xi(t)|$, defined as $|\overline{\xi}(t)| := |\hat{\xi}(t)| + \eta(t)$. This estimate is needed because the controller does not know the current value of $|\xi(t)|$. Note that the estimate satisfies $|\overline{\xi}(t)| = |\hat{\xi}(t)| + \eta(t) \leq |\xi(t)| + 2\eta(t)$. τ_c is a pre-designed parameter specifying when to check for threshold updates. This periodic checking avoids the need for the sensor nodes to continuously listen to the channel, so as to reduce the large amount of energy consumption spent on listening. When there is a threshold update event, the global threshold is updated as

$$\begin{aligned} &\eta(t^c_{r_c+1}) = \mu\eta(t^c_{r_c}) \\ &\eta(t_0) \geq \mu\rho^{-1}\overline{\alpha}^{-1} \circ V(\xi(t_0)), \end{aligned} \tag{14}$$

where $\mu \in (0, 1)$ is a pre-designed parameter. With this new global threshold, each sensor node can compute its current local threshold.

When there is an event from either event-triggered mechanism, only one bit is required for each transmission, whose transmission indicates the event. Furthermore, for the sampling update mechanism, the content of this bit indicates the sign of the error.

This AETC can largely reduce the amount of transmissions, the length of each transmission, and the listening time of each sensor node. However, it still requires a remarkable amount of energy, because of the existence of the extra threshold update mechanism: the sensor nodes still need to wake up and listen to the channel periodically.

Now we state the main problem we solve in this paper:

Problem 1. *Design an autonomous time-varying threshold update mechanism for $\eta(t)$ such that the system (1), (3), (4), (11), and (12) is UGES with λ a lower bound on the decay rate.*

IV. Main result

Consider a Lyapunov function for system (1), (3), (4), (5), (11), and (12) with the following form:

$$V(t) := x^{\mathrm{T}}Px. \tag{15}$$

Note that, for this linear system and the Lyapunov function V defined in (15), Definition 4 applies with scalars $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{R}^+$, such that, the following inequations hold:

$$\begin{aligned} a_1|x|^2 \leq V(t) \leq a_2|x|^2 \\ \dot{V}(t) \leq -a_3V(t) + a_4|e|^2. \end{aligned} \tag{16}$$

For system (1), (3), (4), (5), (11), (12), and (15), an autonomous time-varying threshold update mechanism is designed as:

$$\begin{aligned} &\dot{\eta}(t) = -\lambda\eta(t) \\ &\eta(0) = \sqrt{\omega\frac{a_3 - 2\lambda}{a_4}V(0)}, \end{aligned} \tag{17}$$

where $\omega \in]0, 1]$ a pre-designed parameter, $\lambda :< \frac{a_3}{2}$. Design a reference function $\tilde{V}(t)$ satisfying:

$$\begin{aligned} &\dot{\tilde{V}}(t) = -2\lambda\tilde{V}(t) \\ &\tilde{V}(0) = V(0). \end{aligned} \tag{18}$$

Lemma IV.1. *Consider the system (1), (3), (4), (5), (11), (12), and (15), with threshold update mechanism (17), and reference function $\tilde{V}(t)$ defined in (18), $\forall t \in \mathbb{R}^+_0$, $V(t) \leq \tilde{V}(t)$.*

Proof. According to (11), $\forall t \in \mathbb{R}^+_0$, $|\varepsilon(t)| \leq \eta(t)$. Therefore, (16) can be rewritten by:

$$\dot{V}(t) \leq -a_3V(t) + a_4\eta^2(t). \tag{19}$$

By integrating $V(t)$ on interval $[0, T]$, one obtains:

$$V(T) \leq \frac{\int_0^T e^{a_3t}a_4\eta^2(t)dt + V(0)}{e^{a_3T}}. \tag{20}$$

From (17), $\eta(t)$ can be expressed by:

$$\eta(t) = e^{-\lambda t}\eta(0).$$

Put $\eta(t)$ back in (20) to obtain:

$$\begin{aligned} V(T) &\leq \frac{a_4\eta^2(0)\int_0^T e^{(a_3-2\lambda)t}dt + V(0)}{e^{a_3T}} \\ &= \frac{a_4\eta^2(0)\frac{1}{a_3-2\lambda}\left(e^{(a_3-2\lambda)T} - 1\right) + V(0)}{e^{a_3T}}. \end{aligned} \tag{21}$$

By equation (18), one has:

$$\tilde{V}(T) = e^{-2\lambda T}V(0).$$

We can now bound:

$$\begin{aligned} &V(T) - \tilde{V}(T) \\ \leq &\frac{\frac{a_4\eta^2(0)}{a_3-2\lambda}\left(e^{(a_3-2\lambda)T} - 1\right) + V(0)}{e^{a_3T}} - e^{-2\lambda T}V(0) \\ = &\left(\frac{a_4\eta^2(0)}{a_3 - 2\lambda} - V(0)\right)e^{-a_3T}\left(e^{(a_3-2\lambda)T} - 1\right). \end{aligned}$$

From (17), one has $\frac{a_4\eta^2(0)}{a_3-2\lambda} \leq V(0)$; from $0 < 2\lambda < a_3$, $e^{-a_3T}\left(e^{(a_3-2\lambda)T} - 1\right) > 0$. Therefore, one can conclude that

$$V(t) \leq \tilde{V}(t),\ \forall t \in \mathbb{R}^+_0,$$

which ends the proof. $\square$

With Lemma IV.1, now we can state the main result of this paper:

Theorem IV.2. *The system (1), (3), (4), (5), (11), (12), (15) with threshold update mechanism (17) is UGES with λ a lower bound on the decay rate.*

Proof. From the result of Lemma IV.1, $\forall t \geq 0$, $V(t) \leq \tilde{V}(t) = e^{-2\lambda t} V(0)$. Together with (16), which shows that $a_1|x|^2 \leq V(t) \leq a_2|x|^2$, one concludes that:

$$|\xi(t)| \leq \sqrt{\frac{a_2}{a_1}} e^{-\lambda t} |\xi(0)|.$$

According to Definition 2, the system is GES with convergence rate λ.

To finish the proof, one also needs to prove that, the system is Zeno free. Define $\kappa := \frac{|\xi(t)|}{\eta(t)}$. According to [6], once $\kappa < \infty$ is shown, the system's Zeno freeness can be proven.

$$\begin{aligned}\kappa = \frac{|\xi(t)|}{\eta(t)} &\leq \frac{\sqrt{a_1^{-1} V(t)}}{\eta(t)} \leq \frac{\sqrt{e^{-2\lambda t} V(0)}}{\sqrt{a_1} e^{-\lambda t} \eta(0)} \\ &= \frac{\sqrt{V(0)}}{\sqrt{a_1}\eta(0)} = \sqrt{\frac{a_4}{a_1 \omega (a_3 - 2\lambda)}} < \infty,\end{aligned}$$

where the last inequality comes from (17). This ends the proof. $\square$

Remark. *With the designed threshold update mechanism (17), the computation of $\eta(t)$ only depends on t and the initial value $\eta(0)$. Therefore, once all the nodes have their local clocks synchronised, the initial $\eta(0)$ pre-stored, the global threshold can be computed locally. Thus, the threshold update transmission in AETC from [6] is not necessary.*

V. Initial threshold

From the sampling event condition (11), it is easy to see that a bigger $\eta(t)$ increases the inter-event intervals. As a result, less transmissions are required and less energy is consumed. Therefore, to find the maximum $\eta(0)$ for the threshold update mechanism is important and necessary.

For the Lyapunov function given by (15), define $A_c = A + BK$, $B_c = BK$, $Q = -(A_c^{\mathrm{T}} P + P A_c)$. Consider the following bound:

$$\begin{aligned}\dot{V}(t) &= \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} \\ &= (A_c x + B_c e)^{\mathrm{T}} P x + x^{\mathrm{T}} P (A_c x + B_c e) \\ &= x^{\mathrm{T}} (A_c^{\mathrm{T}} P + P A_c) x + 2 x^{\mathrm{T}} P B_c e \\ &= -x^{\mathrm{T}} Q x + 2 x^{\mathrm{T}} P B_c e \\ &\leq -\lambda_{\min}(Q)|x|^2 + 2|P B_c||x||e|.\end{aligned} \tag{22}$$

Define a_5 and a_6 satisfying $a_5 a_6 = |P B_c|$. Therefore:

$$\begin{aligned}\dot{V}(t) &\leq -\lambda_{\min}(Q)|x|^2 + 2 a_5 a_6 |x||e| \\ &\leq -\lambda_{\min}(Q)|x|^2 + a_5^2|x|^2 + a_6^2|e|^2 \\ &= -(\lambda_{\min}(Q) - a_5^2)|x|^2 + a_6^2|e|^2.\end{aligned} \tag{23}$$

Note that, also for the Lyapunov function (15):

$$\lambda_{\min}(P)|x|^2 \leq V(t) \leq \lambda_{\max}(P)|x|^2.$$

Therefore, $\dot{V}(t)$ is further bounded by:

$$\dot{V}(t) \leq -\frac{\lambda_{\min}(Q) - a_5^2}{\lambda_{\max}(P)} V(t) + a_6^2 |e|^2. \tag{24}$$

From which:

$$a_3 = \frac{\lambda_{\min}(Q) - a_5^2}{\lambda_{\max}(P)},\ a_4 = a_6^2.$$

Selecting suitable a_5 and a_6 determines a_3 and a_4.

By letting $\omega = 1$, one can obtain the maximum $\eta(0)$, defined by $\bar{\eta}(0)$, as:

$$\bar{\eta}(0) := \sqrt{\omega \frac{\frac{\lambda_{\min}(Q) - a_5^2}{\lambda_{\max}(P)} - 2\lambda}{a_6^2} V(0)}. \tag{25}$$

However, from the computation of a_3 and a_4, we can see that $\bar{\eta}(0)$ is in general very conservative.

Inspired by [3], an LMI is designed to compute $\bar{\eta}(0)$. Define $\sigma := \frac{\bar{\eta}(0)}{|\xi(0)|}$. Note that, from now on, $\bar{\eta}(0)$ is computed from $\xi(0)$ directly, instead of from $V(0)$ as suggested by (17).

From (22), we have:

$$\dot{V}(t) = x^{\mathrm{T}} (A_c^{\mathrm{T}} P + P A_c) x + 2 x^{\mathrm{T}} P B_c e,$$

to guarantee $\dot{V}(0) \leq -2\lambda V(0)$, one can impose the following inequation:

$$\begin{aligned}\xi^{\mathrm{T}}(0)(A_c^{\mathrm{T}} P + P A_c)\xi(0) + 2\xi^{\mathrm{T}}(0) P B_c \varepsilon(0) \\ \leq -2\lambda \xi^{\mathrm{T}}(0) P \xi(0).\end{aligned}$$

Rearranging terms one obtains:

$$\xi^{\mathrm{T}}(0)(-Q + 2\lambda P)\xi(0) + 2\xi^{\mathrm{T}}(0) P B_c \varepsilon(0) \leq 0. \tag{26}$$

If Assumption 1 holds, then:

$$|\varepsilon(0)| \leq \eta(0) \leq \bar{\eta}(0) = \sigma |\xi(0)|. \tag{27}$$

Applying the S-procedure, if $\exists \epsilon > 0$ such that:

$$\begin{bmatrix} -Q + 2\lambda P & P B_c \\ B_c^{\mathrm{T}} P & 0 \end{bmatrix} + \epsilon \begin{bmatrix} \sigma^2 I & 0 \\ 0 & -I \end{bmatrix} \preceq 0$$

holds, then (27) implies (26).

To find $\bar{\eta}(0)$, one can instead find the maximum σ by solving the following optimization problem:

$$\begin{aligned}&\max \quad \sigma \\ &\text{subject to} \begin{bmatrix} -Q + 2\lambda P + \epsilon\sigma^2 I & P B_c \\ B_c^{\mathrm{T}} P & -\epsilon I \end{bmatrix} \preceq 0 \\ &\qquad \epsilon > 0, P = P^{\mathrm{T}} \succ 0,\end{aligned} \tag{28}$$

where I is an identity matrix of adequate dimensions.

By now, one can see that, for this mix-triggered strategy to guarantee exponential stability, $\eta(0)$ is a key parameter. The computation of $\eta(0)$ depends on the system model.

If the system model is inaccurate, or not available, one can instead employ the alternative threshold update mechanism:

$$\begin{aligned} \dot{\eta}(t) &= -\lambda\eta(t) \\ \eta(0) &\in]0, \infty[. \end{aligned} \tag{29}$$

Compared with (17), $\eta(0)$ in (29) is an arbitrary scalar. Define:

$$c_2 := \frac{a_4\eta^2(0)}{(a_3 - 2\lambda)V(0)}, \tag{30}$$

where $\lambda :> 0$ is a design parameter, that should be tuned satisfying $\lambda < \frac{a_3}{2}$. Note that a sufficiently small λ can always satisfy this condition. Now, one can rewrite (30) as:

$$\eta(0) = \sqrt{c_2\frac{a_3 - 2\lambda}{a_4}V(0)}. \tag{31}$$

Consider the following reference function:

$$\begin{aligned} \dot{\bar{V}}(t) &= -2\lambda\bar{V}(t) \\ \bar{V}(0) &= \max\{1, c_2\}V(0). \end{aligned} \tag{32}$$

Lemma V.1. *Consider the system (1), (3), (4), (5), (11), (12), and (15), with threshold update mechanism (29), and reference function $\bar{V}(t)$ defined in (32), $\forall t \in \mathbb{R}_0^+$, $V(t) \leq \bar{V}(t)$.*

Proof. Consider $c_2 > 1$ here, since $c_2 \leq 1$ has already been shown in Lemma IV.1. For $V(T)$, it is easy to deduce that:

$$\begin{aligned} V(T) &\leq \frac{a_4\eta^2(0)\frac{1}{a_3-2\lambda}\left(e^{(a_3-2\lambda)T} - 1\right) + V(0)}{e^{a_3T}} \\ &< \frac{a_4\eta^2(0)\frac{1}{a_3-2\lambda}\left(e^{(a_3-2\lambda)T} - 1\right) + c_2V(0)}{e^{a_3T}}. \end{aligned}$$

From (32), one can deduce that:

$$\bar{V}(T) = e^{-2\lambda T}\bar{V}(0) = e^{-2\lambda T}c_2V(0).$$

Therefore:

$$\begin{aligned} &V(T) - \bar{V}(T) \\ &< \left(\frac{a_4\eta^2(0)}{a_3 - 2\lambda} - c_2V(0)\right)e^{-a_3T}\left(e^{(a_3-2\lambda)T} - 1\right). \end{aligned}$$

Since $\frac{a_4\eta^2(0)}{a_3-2\lambda} - c_2V(0) = 0$ by definition, one can conclude that:

$$V(t) < \bar{V}(t) = e^{-2\lambda t}c_2V(0), \forall t \in \mathbb{R}_0^+.$$

This ends the proof. □

Theorem V.2. *The system (1), (3), (4), (5), (11), (12), (15) with threshold update mechanism (29) is GES with λ a lower bound on the decay rate..*

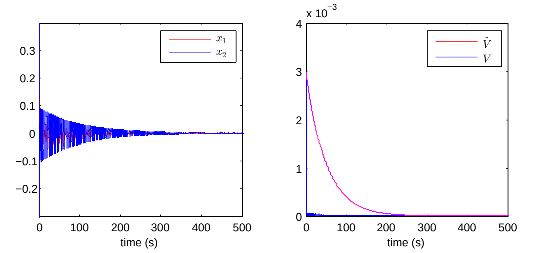

Fig. 2. $\eta(0) = \bar{\eta}(0)$, system state; Lyapunov function evolution and its reference.

Proof. Following the procedure in Theorem IV.2 together with the result in Lemma V.1, one concludes that:

$$|\xi(t)| \leq \max\{1, \sqrt{c_2}\}\sqrt{\frac{a_2}{a_1}e^{-\lambda t}}|\xi(0)|.$$

Which ends the proof. □

Remark. *Theorem V.2 shows that, with threshold update mechanism (29), the system is still GES. However, since c_2 is unknown and dependent on $\xi(0)$, one cannot obtain a uniform bound on the state of the form $|\xi(t)| \leq c|\xi(0)|e^{-\lambda t}$.*

VI. Numerical Example

We illustrate the presented mechanisms in this paper in a system from [10]. In this system, the matrix A, B and K are given by:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K = \begin{bmatrix} 1 & -4 \end{bmatrix}.$$

The system initial state is given as $\xi(0) = \begin{bmatrix} 0.5 & 0.4 \end{bmatrix}^{\mathrm{T}}$; and the performance parameter λ is given as $\lambda = 0.01$. By solving (25), we obtain $\bar{\eta}(0) = 0.0666$. By solving the LMI in (28), we obtain $\sigma = 0.1617$, with:

$$P = \begin{bmatrix} 0.0058 & 0.0016 \\ 0.0016 & 0.0058 \end{bmatrix}.$$

Thus, we obtain $\bar{\eta}(0) = 0.1035$. One can easily see that (25) is very conservative.

Design $\eta(0) = \bar{\eta}(0) = 0.1035$ to reduce transmission and energy consumption. The parameter θ is selected as $\theta = \begin{bmatrix} 0.2357 & 0.9718 \end{bmatrix}^{\mathrm{T}}$. Fig. 2 shows the evolution of the asynchronous mix-triggered implementation's system state $\xi(t)$, Lyapunov function $V(t)$ and its reference $\tilde{V}(t)$. One can see that $V(t) \leq \tilde{V}(t)$, $\forall t \in \mathbb{R}_0^+$, as designed.

Now assume the system model is unknown, we test $\eta(0) = 0.1\bar{\eta}(0)$, $\eta(0) = \bar{\eta}(0)$, $\eta(0) = 10\bar{\eta}(0)$, and $\eta(0) = 100\bar{\eta}(0)$ separately. Also note that, θ is selected as $\theta = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{\mathrm{T}}$. The simulation results are shown in Fig. 3. One can see that for all 4 cases, $V(t) \leq \bar{V}(t)$, $\forall t \in \mathbb{R}_0^+$, as designed. However, the $\bar{V}(t)$ in 4

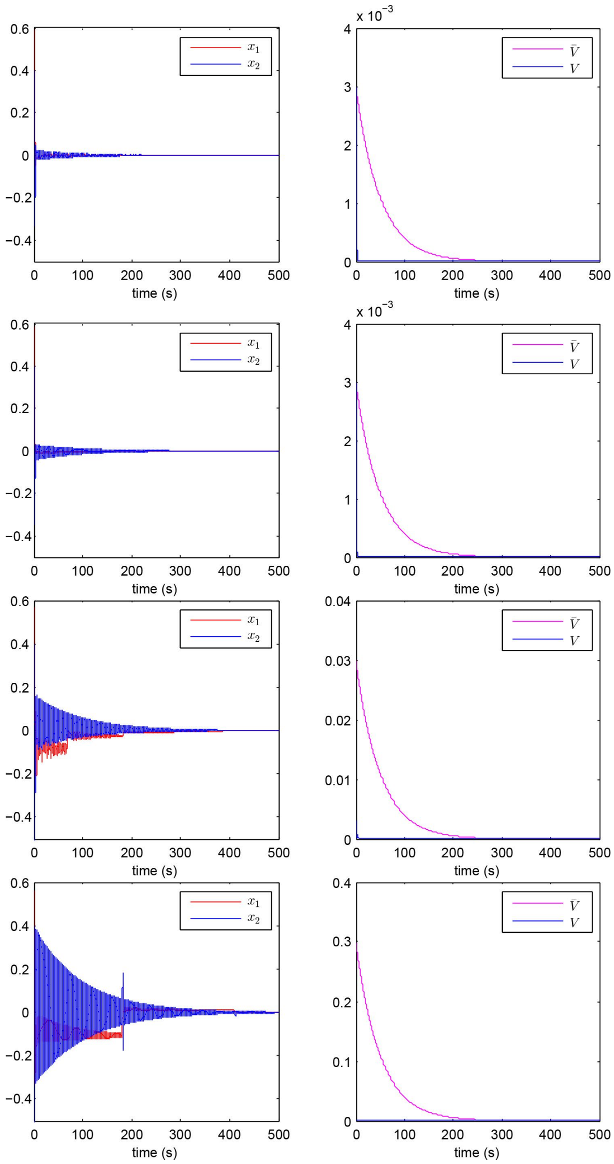

Fig. 3. $\eta(0) = 0.1\bar{\eta}(0)$, $\eta(0) = \bar{\eta}(0)$, $\eta(0) = 10\bar{\eta}(0)$, and $\eta(0) = 100\bar{\eta}(0)$, system states; Lyapunov function evolutions and their references.

cases are varying. We also compare the transmissions required for the state to reach a certain accuracy with different $\eta(0)$. We analyze the number of transmissions for different selections of $\eta(0)$ in the period of time from initialization with $\xi(0) = \begin{bmatrix}0.5 & 0.4\end{bmatrix}^{\mathrm{T}}$ until the state enters a ball around the origin $|\xi(t)| \leq 10^{-5}$. Further simulations show that, when $\eta(0) = 0.1\bar{\eta}(0)$, entering the ball of radius 10^{-5} requires $204.8s$ and 438 events; when $\eta(0) = \bar{\eta}(0)$, it requires $253.2s$ and 407 events; when $\eta(0) = 10\bar{\eta}(0)$, it requires $388.3s$ and 502 events; and when $\eta(0) = 100\bar{\eta}(0)$, it requires $480.6s$ and 771 events. One can see that when $\eta(0) \leq \bar{\eta}(0)$, a smaller $\eta(0)$ leads to a faster convergence rate, but more events. This is in accordance with our statement in the beginning of Section V. When $\eta(0) > \bar{\eta}(0)$, a bigger $\eta(0)$ results in longer convergence time, and requires more events. It seems like selecting exactly $\eta(0) = \bar{\eta}(0)$ optimises the number of events to enter a certain set, but we have no formal proof of that or guarantee that that would always be the case, so more research would be needed.

VII. Conclusion

We have presented an asynchronous mix-triggered control strategy. This strategy is an extension of AETC, with the original event-triggered threshold update mechanism replaced by an autonomous time-varying one. Thus, the transmission of the threshold update signal is not required any more, and the sensors are no longer required to listen to the channel. As a result, the energy of the sensor nodes can be saved. Meanwhile, an LMI is constructed employing the S-procedure to find the maximal initial threshold to guarantee UGES. We also show that the system can still be GES, with uncertain but bounded initial threshold. All these results are validated with a numerical example. Our future work includes considering systems with disturbances, so as to make the work more practical.

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