Convergence of Ant Colony Multi-Agent Swarms

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ABSTRACT

Ant Colony algorithms are a set of biologically inspired algorithms used commonly to solve distributed optimization problems. Convergence has been proven in the context of optimization processes, but these proofs are not applicable in the framework of robotic control. In order to use Ant Colony algorithms to control robotic swarms, we present in this work more general results that prove asymptotic convergence of a multi-agent Ant Colony swarm moving in a weighted graph.

CCS CONCEPTS

• **Mathematics of computing** → *Stochastic processes*; • **Computing methodologies** → *Multi-agent systems*; • **Theory of computation** → Multi-agent learning;

KEYWORDS

swarm robotics, ant colony, random walk, convergence

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1 INTRODUCTION

Decentralised and distributed algorithms have been used largely to solve problems where a divide and conquer approach provides an advantage in either complexity or resource consumption. In recent years, this concept has been applied to robotics in the form of multi-agent systems or swarm robotics. These strategies have clear advantages when solving certain optimisation problems where solutions are constructed piecewise, or in problems where exploration is a key element. For the latter, the advantage of using swarms is clear, but it adds the necessity to analyse the convergence when exploration is no longer needed. When implementing swarm coordination in multi-agent systems, there has been a clear tendency to draw inspiration from nature. Swarms occur naturally in many insect and animal species, therefore attempting to model these biological swarming behaviours is a big part of biomimicry research [3, 11]. In this framework, Ant Colony (AC) algorithms are a subset of biologically inspired stochastic algorithms based on the behavioural traits of ants, used commonly for optimization problems. Their main characteristic is the use of stigmergy: the environment is the main communication medium and information storage tool [5–7]. The agents mark the environment and make stochastic decisions based on the marks they encounter. These algorithms can be used as a control strategy for robotic swarms, either as a path planning system [2, 8, 10, 30] or to directly stablish coordination in a robotic swarm [1, 9, 15, 24].

Consider a foraging and exploitation problem where we need to find a goal in an unknown space and find the shortest route back and forth from the source to the goal. In this unknown space we do not have access to a global positioning system. This is often the case in exploration or mapping problems [28]. The advantages of a stigmergy-based method become more evident: a multi-agent system governed by AC algorithms explores and builds paths simultaneously without the need of centralised instructions. Furthermore, AC algorithms are stochastic in the agent decision process. That is, the agents make choices based on a probability distribution influenced by the marking in the environment they find themselves. This can lead to congestion advantages in traffic routing applications[26]. For both exploration and routing problems, it is common to model the environment as a dynamic weighted graph, with agents moving from vertex to vertex.

With these applications in mind, we are interested in studying the convergence properties of AC algorithms when applied to swarm coordination. Random walks have been largely studied [19, 22, 31], and edge-reinforced random walks on weighted graphs and its asymptotic behaviour has been studied for continuous and discrete time [20, 21, 27]. Alternatively, convergence has been proven for certain kinds of AC optimization algorithms [13, 14, 29], but the results

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do not apply directly to swarm control. In most cases they require of a central entity to analyse all paths the agents are generating, and add more or less weight depending on a cost function. Furthermore, when applying these algorithms to cyber-physical swarms, interacting with the environment translates into some kind of data transmission. In such networks there can be communication restrictions (desired or undesired), under which the existing convergence proofs would not hold. We are interested in applying these techniques to control and route real swarms, hence the motivation to find more general convergence conditions.

The goal of this work is to study the asymptotic convergence of the swarm to a certain distribution, while providing conditions in graph structure and parameter choice. We also give estimates on convergence rates, and how they relate to problem parameters and graph topology. For this, we model the agent environment as a weighted graph, where agents add weight to the edges as they traverse them. The agents have a starting set in which they are initialized, and a target set that they want to reach. When the graph weights are modified by the agent movements, it introduces a time dependency and coupling in the system between the agent and weight dynamics that may give rise to non-Markovian processes. Therefore we make use in this work of results concerning convergence of stochastic matrices of Kushner [16–18] and more recently by Qin et. al. [23] to show under which restrictions in graph structure we still maintain convergence properties, splitting the results for directed and undirected graphs. This is motivated by its different applications. In exploration-exploitation swarm problems undirected graphs are necessary since the exploration of a physical space must be independent of the directionality of the discretization. Alternatively, traffic routing problems require modeling the space with directed graphs, since traffic is directional. The different convergence results are then presented in relation with the different conditions that the system needs to fulfil. At last, we include simulations of a set of standard scenarios to illustrate the convergence results.

2 PRELIMINARIES

2.1 Notation

We denote sets with calligraphic letters and functions with non-calligraphic letters. Vectors are represented in bold. A set whose elements depend on a parameter is indicated as $\mathcal{S}(\cdot)$. Sequences are represented as $\{A(t)\} \equiv \{A_t\} \equiv \{A(0), A(1), ..., A(t)\}$. We consider only discrete time systems, i.e. $t \in \mathbb{N}_0^+$.

2.2 Weighted Graphs

We make use in this work of connected planar graphs, since we aim to represent a 2D geometric space. Therefore, we are free to discretise our space in a graph that is both connected and planar. Furthermore, we consider both directed and undirected graphs. We refer to an edge connecting i to j as $\{ij\} \equiv \{ji\}$ if the graph is undirected, and (ij) if the graph is directed. For simplicity, all concepts and definitions regarding weighted graphs will be define using undirected notation (edge from i to j as $\{ij\}$), but will apply to both directed and undirected graphs unless the opposite is stated.

Definition 1. *We define a time-varying weighted graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, W(t))$ as a tuple including a vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and weights $W : \mathbb{N}_0^+ \to \mathbb{R}_+^{|\mathcal{V}|\times|\mathcal{V}|}$, where each value $W_{ij}(t)$ is the weight assigned to edge $\{ij\} \in \mathcal{E}$. Furthermore, the graph is connected if for every pair $i, j \in \mathcal{V}$ there exists a set of edges*

$$\{\{iu_1\}, \{u_1u_2\}, \dots, \{u_nj\}\} \subseteq \mathcal{E}$$

that connects i and j.

The image of a function assigning values to edges in a graph can be written as a matrix, and the subscript will indicate both edges and entries in the image of the function. That is, let $f : \mathbb{N} \to \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$. Then, $f_{ij}(k)$ is the ij-th entry in the image $f(k)$, which corresponds to the edge $e \equiv \{ij\}$. We use this function class for the graph weights, and by definition

$$W_{ij}(\cdot) := 0 \quad \forall \{ij\} \notin \mathcal{E}.$$

The degree of vertex i is $d_i := |\{\{ij\} : \{ij\} \in \mathcal{E}, j \in \mathcal{V}\}|$, and weighted degree is

$$w_i(t) := \sum_{k \in \mathcal{V}} W_{ik}(t).$$

Furthermore, when considering directed graphs the degree d_i refers to the out-degree unless the opposite is stated. For undirected graphs $W_{ij}(t) \equiv W_{ji}(t)$, but the converse does not necessarily hold for directed graphs.

Definition 2. *[4] An i−j path in $\mathcal{G}$ is a subgraph $\mathcal{V}' \subseteq \mathcal{V}$, $\mathcal{E}' \subseteq \mathcal{E}$*

$$\mathcal{V}' = \{i, k, l, ..., z, j\},\ \mathcal{E}' = \{\{ik\}, \{kl\}, ..., \{zj\}\}$$

where no vertex appears twice. An i-cycle is then a closed path $i - i$ starting and ending in the same vertex $i \in \mathcal{V}$.

The diameter of the graph δ is the length of the longest path for any $i, j \in \mathcal{V}$.

Definition 3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be any connected graph. We define the frontier of a subset of vertices $\mathcal{K}_1 \subset \mathcal{V}$ with respect to a second subset $\mathcal{K}_2 \subset \mathcal{V}$ where $\mathcal{K}_1 \cap \mathcal{K}_2 = \varnothing$ as*

$$F(\mathcal{K}_1 \to \mathcal{K}_2) := \{u \in \mathcal{K}_1 : \{vu\} \in \mathcal{E}, v \in \mathcal{K}_2\}.$$

2.3 Stochastic Matrices and Convergence

The expected value of a random variable X is denoted as $E[X]$, and when conditioned to a sequence Y_t we denote $E[X \mid Y_t]$.

Definition 4. *[12] A sequence of integrable random variables X_t measurable with respect to a sequence of increasing $\sigma-$algebras $\{\mathcal{F}_t\}$ is called a Martingale if*

$$E[X_{t+1}|\mathcal{F}_t] = X_t \quad a.s. \quad \forall t \geqslant 0,$$

When considering a discrete time system, $\{\mathcal{F}_t\}$ includes all the information until time t.

A stochastic matrix is a square matrix $P \in \mathbb{R}^{n\times n}$ with non-negative entries and the sum of its rows (or columns) each equal to 1. Their use to represent Markovian processes has been extensively studied, since the probability transition matrix of a Markovian discrete time process can be represented with such matrices. It is useful to define the following stochastic convergence concept.

Definition 5 (Almost Sure Convergence[12]). *Let Ω be a probability sample space, with $\omega \in \Omega$ being any event. We say a sequence of random variables $X_0, X_1, ..., X_t$ converges almost surely (a.s.) to a random variable X_∞ as $t \to \infty$ iff*

$$\Pr\left[\{\omega : X_t(\omega) \to X_\infty \text{ as } t \to \infty\}\right] = 1.$$

In this work we make use of convergence results for the product of stochastic matrices presented by Qin et. al. [23]. For this purpose, we introduce the following concepts presented in their work. Let $\mathcal{M}_2$ be the class of all scrambling matrices (no two rows are orthogonal)[25].

Assumption 1 (Qin et. al. [23]). Let $A(t)$ be a discrete time dependent row stochastic matrix, with $\prod_{t=j}^{t=k} A(t)$ its left product from k to j (i.e. $A(k)A(k-1)A(k-2)...A(j)$). Suppose the process satisfies:

(1) There exists integer $h > 0$ such that for all $k > 0$:

$$\Pr\left[\prod_{t=k}^{h+k} A(t) \in \mathcal{M}_2\right] > 0,$$

$$\sum_{i=1}^{\infty} \Pr\left[\prod_{t=k+(i-1)h}^{k+ih} A(t) \in \mathcal{M}_2\right] = \infty.$$

(2) There is a positive α such that any $A_{ij}(t) > \alpha$ if $A_{ij}(t) > 0$.

Theorem 1 (Qin et. al. [23]). *Under Assumption 1, the product of the sequence of row stochastic matrices $\prod_{t=0}^{t=k} A(t)$ converges to a random matrix of identical rows $L = \mathbf{1}\xi^T$ a.s. as $k \to \infty$, where $\xi \in \mathbb{R}^n$ satisfies $\xi^T\mathbf{1} = 1$.*

Note that the results in Theorem 1 do not imply that the stochastic matrix $A(t)$ converges, only its product. We show in the following Corollary that the same result applies for column stochastic matrices, but with convergence to L^T.

Corollary 1. *The results in Theorem 1 apply similarly to a sequence of column stochastic matrices. In particular, for a sequence $\{B_t\}$ where $B_i \in \mathbb{R}^{n\times n}$ and all B_i^T satisfy Assumption 1:*

$$\lim_{k\to\infty} \prod_{t=0}^{t=k} B(t) = \left(\mathbf{1}\xi^T\right)^T$$

Proof. We can show this by contradiction. The results presented in Theorem 1 are formulated for row-stochastic matrices. Let $\mathcal{M}$ be a class of row stochastic matrices which sequences satisfy Assumption 1. Consider the sequence of column stochastic matrices $\{B_k\}$, with any sequence formed by transposed elements $\{B_i^T\} \in \mathcal{M}$. Let $\mathcal{M}_B$ be the set of all possible matrices B_i, and $\mathcal{M}_{B^T}$ the set of all B_i^T, such that $B_i \in \mathcal{M}_B$ and $B_i^T \in \mathcal{M}_{B^T}$ for any i. Consider the left product of the original sequence. Observe that we can take the transposed of the product:

$$\left[\prod_{t=0}^{t=k} B_t\right]^T = \prod_{t=0}^{t=k} A_t, \tag{1}$$

where $A_t \in \mathcal{M}_{B^T}$ for all t. If the limit as $k \to \infty$ of (1) does not exist, there exists a sequence $\{A_t\}$ for which its product does not converge. But by definition, the sequence $\{A_t\}$ satisfies Assumption 1 since $A_i \in \mathcal{M}_{B^T}$, and any sequence $\{B_i^T\} \in \mathcal{M}$. Therefore, the limit in (1) must satisfy (*a.s.*):

$$\lim_{k\to\infty} \prod_{t=0}^{t=k} B_t = \lim_{k\to\infty} \left[\prod_{t=0}^{t=k} A_t\right]^T = \left(\mathbf{1}\xi^T\right)^T,$$

where $\xi \in \mathbb{R}_+^n$ and all its entries sum to 1. □

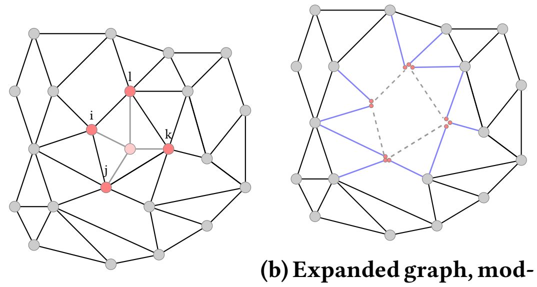
3 SYSTEM DESCRIPTION

Let $\mathcal{G}_0$ be a weighted connected graph as in Definition 1. Let $\mathcal{A} = \{1, 2, ..., n\}$ be a set of agents walking from vertex to vertex. The position of agent a at time t is $x_a(t) = v,\ v \in \mathcal{V}_0$, and we group them in a vector $X(t) := \{x_a(t) : a \in \mathcal{A}\}$. The position of the agents will evolve depending on some probability transition matrix $P(t) : \mathbb{N}_0^+ \to \mathbb{R}^{|\mathcal{V}_0|\times|\mathcal{V}_0|}$.

For certain swarm problems (exploitation, shortest path), agents walk around in the graph trying to find a target set $\mathcal{T}_0 \subset \mathcal{V}_0$ starting from a start set $\mathcal{S}_0 \subset \mathcal{V}_0$. This imitates the behaviour of ants starting at a nest and looking for a food source. This motivates the modification of the graph such that this is reflected in our probability transition matrix.

3.1 Graph Expansion

The starting and target set represent the "ant nest" and "food source" in the biomimicry parallelism. We want the agents to visit these sets infinitely often, but not necessarily staying



(a) Planar graph. (b) Expanded graph, modified edges (blue).

Figure 1: Graph Expansion with target set (red)

in them for more than one time step. After finding one of the sets, the agent must turn around into its previous vertex; if agents find $\mathcal{T}_0$, the best strategy to return to $\mathcal{S}_0$ is to follow their last direction. To solve this and not have a set of agent dependent $P(t)$, we can build an alternative graph in the following way. Consider the graph in Figure 1a. The target set is represented in red vertices, with the frontier set in darker red ($\{i, j, k, l\}$) and the inner vertex in light red. The inner edges are represented in light grey. Since we only need the agents to find any vertex $v \in \mathcal{T}_0$, we can disregard the inner vertices in $\mathcal{T}_0$ when constructing the probability transition matrix. Now consider the following vertex expansion. We take the frontier vertices and eliminate from the graph the edges connecting them, and we divide the vertices into d_i new vertices with degree 1. This transformation on the graph is represented in Figure 1b. Red vertices are the expanded set of target vertices, and blue edges its corresponding edges. See that with this we replace each target set column and row in $P(t)$ by d_i new rows, each with only one entry $p = 1$. With this we can define an extended graph $\mathcal{G}$, with corresponding modified sets $\mathcal{V}, \mathcal{E}, W(t), \mathcal{T}$ and $\mathcal{S}$, starting from a graph $\mathcal{G}_0$.

Definition 6. *Let $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, W_0(t))$ be a connected, planar, weighted graph. Let $\mathcal{T}_0 \subset \mathcal{V}_0$, $\mathcal{S}_0 \subset \mathcal{V}_0$ be a target and starting set, with adjacent edge sets $\mathcal{E}_0^{\mathcal{T}_0}$, $\mathcal{E}_0^{\mathcal{S}_0}$ and frontier sets $\mathcal{T}_0^f = F(\mathcal{T}_0 \to \mathcal{V}_0 \setminus \mathcal{T}_0)$, $\mathcal{S}_0^f = F(\mathcal{S}_0 \to \mathcal{V}_0 \setminus \mathcal{S}_0)$. We define an expanded graph $\mathcal{G}_{\mathcal{T},\mathcal{S}} = (\mathcal{V}, \mathcal{E}, W(t))$ with expanded target and starting sets $\mathcal{T}, \mathcal{S}$. The expanded graph $\mathcal{G}$ is also connected, since we remove interior vertices to connected sub-graphs.*

Remark 1. If the sets $\mathcal{T}_0$ and $\mathcal{S}_0$ are adjacent, the expansion would produce a disconnected graph. Nevertheless, this is a pathological case and therefore it is assumed the minimum distance between both sets is larger than 1 vertex.

For a complete definition and construction of the expanded sets, see Section 7.1. Consider again Figure 1b. It is clear that by taking the expanded form of the graph we are not changing the geometric shape of the graph. It simply eliminates the interior of target and starting sets, and transforms the vertices such the agents turn around by adding rows and columns to $P(t)$. Although it does limit the behaviour of the agents. The agents cannot pass through the original sets $\mathcal{T}_0$, $\mathcal{S}_0$, nor can they walk around the frontiers. However, we consider this behaviour desirable to our problem; when introducing a target and starting sets of more than one vertex, a pathological behaviour would be to stay permanently in one of the two sets. By constructing the expanded graph $\mathcal{G}$ we avoid this behaviour modifying the graph structure.

3.2 Agent Dynamics

In our AC system, we are interested in getting our agents to converge to trajectories connecting a starting set $\mathcal{S}$ and a target set $\mathcal{T}$ infinitely often. First, consider all the vertices in our graph that are not connected to $\mathcal{S}$ nor $\mathcal{T}$. In this case, the agents move by selecting adjacent vertices based on the weight dependent probability distribution

$$\Pr\{x_a(t+1) = j \mid x_a(t) = i\} = \frac{W_{ij}(t)}{w_i(t)}, \ a \in A, \ i, j \notin \mathcal{T} \cup \mathcal{S}. \tag{2}$$

This is analogous to a biased random walk in a graph. Furthermore, operating with the expanded form of the graph as described in Definition 6 enables us to write the probability transition matrix for an expanded weighed graph as

$$P_{ji}(t) = \begin{cases} 1 & \text{if } i \in \mathcal{T}' \cup \mathcal{S}', \ \{ij\} \in \mathcal{E}' \\ \frac{W_{ij}(t)}{w_i(t)} & \text{else.} \end{cases} \tag{3}$$

This translates into the following dynamics for the agent probability distribution.

Definition 7. *The distribution of agents $\mathbf{y} : \mathbb{N}_0^+ \to \mathbb{R}_+^{|\mathcal{V}|}$ is the probability of having an agent in any vertex $i \in \mathcal{V}$ at time t. The distribution evolves according to*

$$\mathbf{y}(t+1) = P(t)\mathbf{y}(t).$$

That is, given a distribution $\mathbf{y}(t)$, the product $P(t)\mathbf{y}(t)$ gives us the distribution at the next time step. Note that P_{ji} represents then the probability of moving from i to j, and $\Pr\{x_a(t+1) = j\}$ is the j-th entry of $\mathbf{y}(t+1)$. The distribution is initialised to some initial distribution $\mathbf{y}(0) = \mathbf{y}_0$.

Remark 2. See that the agent distribution $\mathbf{y}(t)$ follows Markovian dynamics; the probabilities at time $t+1$ are fully determined by the state at t. Although this does not imply the system is fully Markovian; the probability transition matrix may follow an underlying non-Markovian process (this is in fact the case, as showed in the following section).

Observe that now the purpose of the graph expansion procedure becomes clearer. By generating the expanded graph $\mathcal{G}'$ we are able to incorporate an implicit one-step memory while having $P(t)$ not depend on the agent vertex history.

3.3 Graph Dynamics

Let us first define the following agent movement matrix.

Definition 8. *The matrix of agent movements as a function of time, $M : \mathbb{N}_0^+ \to \mathbb{R}^{|V|\times|V|}$, has entries*

$$M_{ij}(t+1) := |\{a \in \mathcal{A} : x_a(t+1) = j\,,\, x_a(t) = i\}|, \quad (4)$$

that is, the entry i, j of the matrix $M(t+1)$ is the amount of agents that were at vertex i at time t, and move to vertex j at time $t+1$.

Observe that $M_{ij}(t+1)$ is a random variable, since it depends on the agent state at $t+1$ and this follows a stochastic process described in Definition 7. With this we can write the weight dynamics in the graph.

Definition 9. *Let M be an agent movement matrix. If $\mathcal{G}$ is a directed graph, each time step the graph weight matrix is updated following the dynamics*

$$W(t+1) = (1-\rho)W(t) + \Delta w M(t+1), \quad (5)$$

where $\rho \in (0,1)$ is a chosen evaporation factor, and $\Delta w = \frac{\rho}{n}$ is the amount of weight each agent adds to the edge. If $\mathcal{G}$ is undirected, M_{ij} and M_{ji} act over the same edge, and the dynamics are

$$W(t+1) = (1-\rho)W(t) + \Delta w \left(M(t+1) + M^T(t+1)\right). \quad (6)$$

All weights are initialised to a uniform weight distribution, $W(0) = \omega_0 A$, where A is the adjacency matrix of $\mathcal{G}$.

The value of n may be limited to the practical application, but in principle Δw is a design parameter and we are free to choose any value. The choice of $\Delta w = \frac{\rho}{n}$ is motivated by the fact that it ensures the total amount of weight will be constant if the initial weight amount adds to 1, i.e. $\sum_i \sum_j W(t)_{ij} = 1\ \forall t > 0$ if $\omega_0|\mathcal{E}| = 1$, both for directed and undirected graphs.

We can now show why the process is not Markovian. The evolution of $P(t)$ depends on the evolution of $W(t)$. If $W(t)$ is fully known, then $P(t)$ is Markovian. But the only way of knowing $W(t)$ is by knowing the entire sequence $M(0), M(1), \ldots, M(t-1)$. Therefore, $P(t)$ does depend on states previous to $t-1$, and it cannot be a Markovian process.

In fact $M(t)$ cannot be considered to be Markovian either. The probability $P\{M(t+1) = M_{t+1} \mid M(t) = M_t, M(t-1) = M_{t-1} \ldots\} \neq P\{M(t+1) = M_{t+1} \mid M(t) = M_t\}$; the probabilities of $M(t)$ taking certain values depend on the underlying graph weight distribution $W(t)$. But $W(t)$ is in fact determined by the entire sequence of movements $M(t), M(t-1), \ldots, M(1)$. Only by knowing the movements $M(t)$ we cannot reconstruct $W(t)$, therefore the values $M(t+1)$ are dependent on the entire sequence $M(1), M(2), \ldots, M(t)$.

3.4 Problem Definition

We consider now the graph and agent dynamics together to define the complete AC Swarm system in a graph.

Definition 10. *We define an AC Graph System $AS := (\mathcal{G}, \{X(t)\}, \Lambda)$ where $\mathcal{G}$ is an expanded weighted, planar connected graph built from a certain $\mathcal{G}_0$ with at least one odd length cycle. The weights $W(t)$ follow the dynamics in Definition 9. The agent positions $\{X(t)\} := \{X(0), X(1), \ldots, X(t)\}$ follow the agent probability distribution dynamics in Definition 7. Finally, $\Lambda := (\mathcal{T}, \mathcal{S}, P(t))$ is the tuple of restrictions to the agent movements, with $P(t)$ defined as (3). The sets $\mathcal{T}, \mathcal{S}$ are the expanded target and starting sets, constructed from some $\mathcal{T}_0, \mathcal{S}_0$.*

Remark 3. Observe the requirement of $\mathcal{G}$ being connected and having at least one odd length cycle. This implies that for long enough times, any vertex $i \in \mathcal{V}$ is reachable from any other $j \in \mathcal{V}$. This is a common concept when studying random walks, and it is shown in the next section. The necessity of this will become clear in further sections.

We are ready now to formulate the convergence problem that concerns this work.

Problem 1. *Let an AC Graph System AS as defined in Definition 10. Can we ensure the distribution of agents around the graph $\mathcal{G}$ converges to a stationary distribution $\mathbf{y}_\infty$? and, what are the conditions for the graph topology and parameters that need to be satisfied?*

4 RESULTS

As pointed out in Definition 9, the weight dynamics are different if we consider a directed graph since the weights $W_{ij}(t)$ are affected by the symmetric agent movements $M_{ji}(t)$. This motivates to approach the problem in slightly different ways for directed or undirected graphs. We first present general convergence results that hold for any connected graph. After that, we present stronger convergence results in the case the graph is directed. The proofs for all the statements in this section are included in Section 7.

4.1 Connected Graphs: y_∞ Convergence

Recall the agent distribution dynamics in Definition 7. With any connected graph, we can write the distribution at any time $t > 0$ as

$$\mathbf{y}(t+1) = P(t)\mathbf{y}(t) = P(t)P(t-1)\mathbf{y}(t-1) = \ldots = \prod_{k=0}^{k=t} P(k)\mathbf{y}(0).$$

Therefore, if the limit $L_\infty := \lim_{t\to\infty} \prod_{k=0}^{k=t} P(k)$ exists,

$$\lim_{t\to\infty} \mathbf{y}(t+1) = \lim_{t\to\infty} \prod_{k=0}^{k=t} P(k)\mathbf{y}(0) = L_\infty \mathbf{y}(0) =: \mathbf{y}_\infty. \quad (7)$$

That is, if we can show the product of our sequence of stochastic matrices $P(t)$ converges to a stochastic matrix, the agent distribution will converge to a stationary distribution. For this, let us defined a restricted weight matrix.

Definition 11. *Let $\mathcal{G}$ be a connected planar graph. We defined a restricted weight matrix $\underline{W}(t)$ constructed from $W(t)$ such that, $\forall \{ij\} \in \mathcal{E}$:*

$$\underline{W}_{ij}(t) = \begin{cases} W_{ij}(t) & \text{if } W_{ij}(t) \geq \varepsilon, \\ \varepsilon & \text{else.} \end{cases}$$

Then, the matrix

$$\underline{P}_{ji}(t) = \begin{cases} 1 & \text{if } i \in \mathcal{T}' \cup \mathcal{S}', \ \{ij\} \in \mathcal{E}' \\ \frac{\underline{W}_{ij}(t)}{\underline{w}_i(t)} & \text{else,} \end{cases}$$

is the restricted probability transition matrix.

To show the agent distribution convergence properties, we first present the property introduced in Remark 3.

Proposition 1. *Let $\mathcal{G}$ be an undirected weighted connected graph. Let $\mathcal{G}$ have at least one odd length cycle C of length l_c. Let p^t_{ij} be the probability of any path reaching vertex j from i in time t. Let δ be the diameter of the graph. Then,*

$$t \geqslant 2\delta + l_c \Rightarrow p^t_{ij} > 0 \quad \forall i, j \in \mathcal{V}.$$

Remark 4. We consider graphs that represent geometric discretisations of space. Since we can always add a self loop in a vertex with weight ε, we consider that effectively the bound in Proposition 1 can be tightened to $t \leq 2\delta + 1$.

Proposition 2. *Let AS be an AC system. Let $\mathcal{G}$ be any connected planar graph. Let a minimum weight $\varepsilon > 0$ such that we can construct the restricted $\underline{W}(t)$, $\underline{P}(t)$. If $\mathcal{G}$ has at least one cycle of odd length, the sequence $\{\underline{P}^T(t)\}$ satisfies the conditions in Assumption 1.*

Now, we present the main result for any connected graph regarding agent distribution convergence.

Theorem 2 (Agent Distribution Convergence). *Let AS be an AC graph system from Definition 10. Let $\mathcal{G}$ be any connected planar graph. If a minimum weight ε is set in every edge, the graph $\mathcal{G}$ will remain connected, and the product $\prod_{t=0}^{t=k} \underline{P}(t)$ converges to a column matrix a.s. as $t \to \infty$,*

$$\lim_{k\to\infty} \prod_{t=0}^{t=k} \underline{P}(t) = \xi \mathbf{1}^T, \tag{8}$$

where $\xi \in \mathbb{R}_+^{|\mathcal{V}|}$ has all entries adding to 1.

Corollary 2. *Let AS be an AC graph system. Let $\mathcal{G}$ be any connected planar graph. Let ε be the minimum weight set in each edge. Let every agent $a \in \mathcal{A}$ use a different weight matrix $W^a(t)$ such that*

$$W^a_{ij}(t) = \begin{cases} 0 & \text{if } X_a(t) = i \text{ and } X_a(t-1) = j, \\ W_{ij}(t) & \text{else.} \end{cases}$$

Then, each agent will converge to a certain stationary distribution $y^a(t) \xrightarrow{a.s.} y^a_\infty$ as $t \to \infty$.

Corollary 3. *Let AS be an AC graph system. Let $\mathcal{G}$ be any connected planar graph. Let ε be the minimum weight set in each edge. Let $\phi^a \in \{0, 1\}$ be a random variable taking value 1 if a communication event from agent $a \in \mathcal{A}$ takes place, and value 0 otherwise. If ϕ^a is independent of $M(t)$, then it does not affect convergence properties of the system.*

4.2 Directed Graphs: P_∞ Convergence

In a directed graph, the weights of an AC graph system AS, and edges (ij) are not affected by the changes in edge (ji). Considering this, to prove the main result for directed graphs we present first a set of necessary concepts.

Proposition 3. *Let $AS = (\mathcal{G}, \{X(t)\}, \Lambda)$ be an AC system. Let its state be fully defined at time t by σ-algebra*

$$\mathcal{F}_t = \sigma(\{M(0), M(1)..., M(t)\}),$$

where $\mathcal{F}_t \subset \mathcal{F}$, and $\mathcal{F}$ is the set of all possible events (combinations of agent choices). At last, let $n_i(t) := |\{a \in \mathcal{A} \mid X_a(t) = i\}|$ be the total amount of agents in vertex i at time t. Then, the position of an agent $X_{a_0}(t+1)$ is a random variable independent of other agent positions $X_{a_k}(t+1)$, $a_k \in \mathcal{A} \setminus \{a_0\}$, and the conditional expected value of $M(t+1)$ is

$$E\left[M_{ij}(t+1) \mid \mathcal{F}_t\right] = P_{ji}(t) n_i(t).$$

Remark 5. The sum over the rows in $M(t+1)$ depends on the state of our system at time t. More specifically,

$$\sum_{j\in\mathcal{V}} M_{ij}(t+1) = |\{a \in \mathcal{A} \mid X_a(t) = i\}| \equiv n_i(t).$$

Similarly, the weighted degree $w_i(t+1)$ is also determined if we know the values of $M(0), M(1), ..., M(t)$. By definition

$$w_i(t+1) = \sum_{k\in\mathcal{V}} (1-\rho) W_{ik}(t) + \frac{\rho}{n} M_{ik}(t+1) =$$

$$=(1-\rho) w_i(t) + \frac{\rho}{n} \sum_{k\in\mathcal{V}} M_{ik}(t+1).$$

Then,

$$w_i(t+1) = (1-\rho) w_i(t) + \frac{\rho}{n} n_i(t).$$

With this, we can show a strong stochastic property of the evolution of $P(t)$ when the underlying graph is directed.

Proposition 4. *Let AS be an AC graph system from Definition 10. Let $\mathcal{G}$ be a directed graph, i.e. $W_{ij} \neq W_{ji}$. Let the increasing σ algebra $\mathcal{F}_t = \sigma(\{M(0), M(1)..., M(t)\})$, where $\mathcal{F}_t \subset \mathcal{F}$, and $\mathcal{F}$ is the set of all possible events (combinations of agent choices). Finally, let the temporal increment in any entry ij of the probability transition matrix $P(t)$ be defined $\Delta P_{ji}(t) := P_{ji}(t+1) - P_{ji}(t)$. For any $\rho \in (0, 1)$,*

$$E\left[\Delta P_{ji}(t) \mid \mathcal{F}_t\right] = 0.$$

We introduce Doob's Martingale convergence Theorem.

THEOREM 3 (DOOB'S MARTINGALE CONVERGENCE [12]). *Let X_n be a Martingale such that*

$$\sup_n E[X_n^+] < \infty.$$

Then, X_n converges a.s..

At last, we present the main Theorem of this section.

THEOREM 4. *[Transition Probability Convergence for directed graphs] Let $AS = (\mathcal{G}, \{X(t)\}, \Lambda)$ be an AC graph system. Let $\mathcal{G}$ be a directed graph with minimum weight $\varepsilon = 0$. Then, the probability transition matrix of the agent movement converges a.s. to a stationary P_∞. That is, $P(t) \xrightarrow{a.s.} P_\infty$ as $t \to \infty$.*

Remark 6. In an undirected graph, the probabilities $P_{ji}(t)$ can be affected by flow of agents moving inwards to i. Theorem 4 relies on the fact that this does not happen to directed graphs. Nevertheless, the authors believe an analogous proof can be established for undirected graphs, using the fact that the edges of the graph are modified by a set of agents that do converge to a fixed distribution.

COROLLARY 4. *Let AS be an AC graph system. Let $\mathcal{G}$ be a directed connected planar graph. Let $\varepsilon = 0$ be the minimum weight set in each edge. Let $\phi^a \in \{0, 1\}$ be a random variable taking value 1 with probability p_ϕ if a communication event from agent $a \in \mathcal{A}$ takes place, and value 0 o.w. If ϕ^a is independent of $M(t)$, then $P(t) \xrightarrow{a.s.} P_\infty$ as $t \to \infty$.*

4.3 Convergence Speed

Consider the results of Theorem 2. By establishing a minimum weight ε we ensure convergence of the agent distribution as $t \to \infty$. Let us recall concepts from Qin et. al. [23].

DEFINITION 12 (QIN ET. AL. [23]). *The sequence $\{W(i)\}$ is said to converge exponentially fast to Y at a rate no slower than γ^{-1} for some $\gamma > 1$ independent of an event ω if $\gamma^k \|W_k - Y\| = Z$ for some $Z \geq 0$.*

THEOREM 5 (QIN ET. AL. [23]). *In addition to Assumption 1, if there exists a number $p \in (0, 1)$ such that for any $k \in \mathbb{N}_0$ we have $\Pr\left[\prod_{i=k}^{h} W(i) \in \mathcal{M}_2\right] \geq p > 0$, then the almost sure convergence of the product to a random matrix L is exponential, and the rate is no slower than $(1 - p\alpha^h)^{1/h}$.*

Remark 7. Recall Proposition 2. By adding a minimum weight ε, the graph is connected for all t and since there exists at least an odd length cycle,

$$\Pr\left[\left(\prod_{t=t_0}^{t_0+2\delta+1} \underline{P}(t)\right)^T \in \mathcal{M}_2\right] = 1 \quad \forall t_0.$$

Therefore, with $p = 1$ and $\alpha = \frac{\varepsilon}{1+(d_i^*-1)\varepsilon}$, the convergence rate for an AS system with minimum weight ε is no slower than $(1 - \alpha^{1+2\delta})^{\frac{1}{1+2\delta}}$.

5 SIMULATIONS

To show the convergence results in simulated examples, we restrict our cases to the following baseline scenarios. First, all edge weights are initialised to a uniform value $W(0) = \omega_0 A$, where A is the adjacency matrix and $\omega_0 = 1/|\mathcal{E}|$.

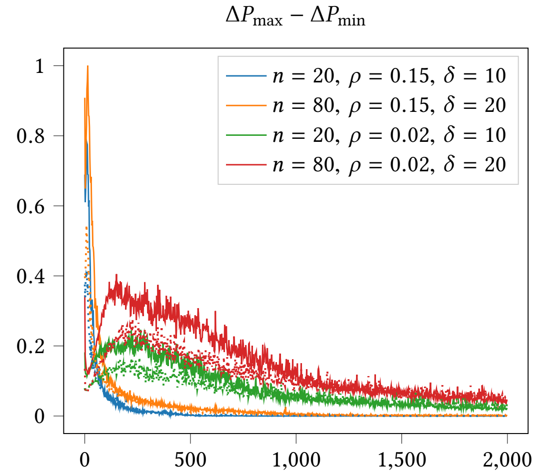
- Directed and undirected triangular planar lattices.
- $|\mathcal{S}| = |\mathcal{T}| = 1$. Sets placed randomly in the graph.
- $\delta \in \{10, 20\}$, $|\mathcal{A}| \in \{20, 80\}$.
- $\varepsilon \in \{0, \frac{\omega_0}{5}\}$, $\rho \in \{2 \cdot 10^{-2}, 1.5 \cdot 10^{-1}\}$

We consider $\varepsilon = 0$ for both directed and undirected graphs. This is since, although we only showed P_∞ convergence for directed graphs, by Remark 6 there is enough reason to believe it will also converge for directed graphs. For simplicity, we consider only triangular planar lattice graphs. Therefore, there is no need to add a self loop in the graph, and $\mathcal{G}$ satisfies the necessary conditions. The choice of low ρ values is motivated by the size of the graphs. The parameter ρ influences how fast weights go to zero (or ε). A value of $\rho = 0.05$ yields a half life time of $t_{1/2} \approx 13$ time steps, and we consider graphs of diameters between 10 and 20.

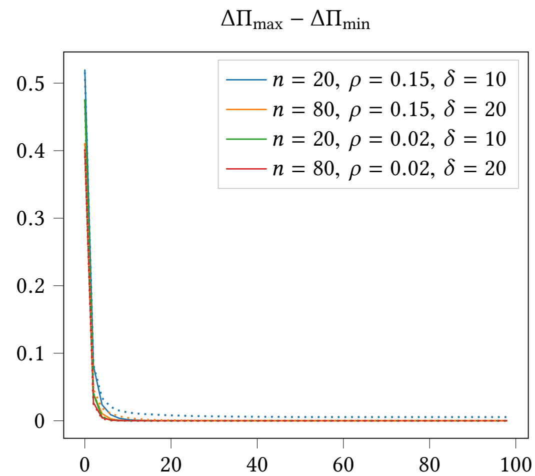
To show the convergence in the case of P_∞ we plot the values $\Delta P_{\max} - \Delta P_{\min}$,where $\Delta P_{\max} = \max_{i,j}\{P(t+1) - P(t)\}$, and the converse for the minimum. To show convergence of the matrix product to an identical column matrix, let first $\Delta\Pi = \left[\prod_{t=0}^{k} P(t)\right]_i - \left[\prod_{t=0}^{k} P(t)\right]_j$, where $\left[\prod_{t=0}^{k} P(t)\right]_i$ is the i-th column of the matrix product, and i and j are chosen at random among all columns. Therefore, to show convergence we plot $\Delta\Pi_{\max} - \Delta\Pi_{\min}$.

5.1 Convergence Results

Figures 2a and 2b show the convergence results both for the matrix $P(t)$ and the product of matrices with $\varepsilon = 0$, and Figure 3 shows the convergence of the product for $\varepsilon = \omega_0/5$. Each line represents the average of 50 simulations done with the same parameter set. The colors correspond to a fixed set of parameter in the legend, dotted lines are undirected graphs and full lines directed graphs. Note from Figure 2b how the convergence of the matrix product is indeed exponential, and has a very fast convergence rate. However, from Theorem 4, we require the minimum weight to be set to zero to ensure the convergence of $P(t)$, but $\varepsilon > 0$ to have convergence in the matrix product. From Figures 2a and 2b we can see that both the matrix product and $P(t)$ converge. This is consistent with the results in Theorems 4 and 2; for convergence to P_∞ we need to set $\varepsilon = 0$. By setting $\varepsilon = 0$ we allow the graph to become virtually disconnected, therefore in some



(a) ***P*** **matrix convergence.**



(b) $\prod_{t=0}^{k} P(t)$ **convergence.**

Figure 2: Directed and Undirected Graphs with $\varepsilon = 0$**.**

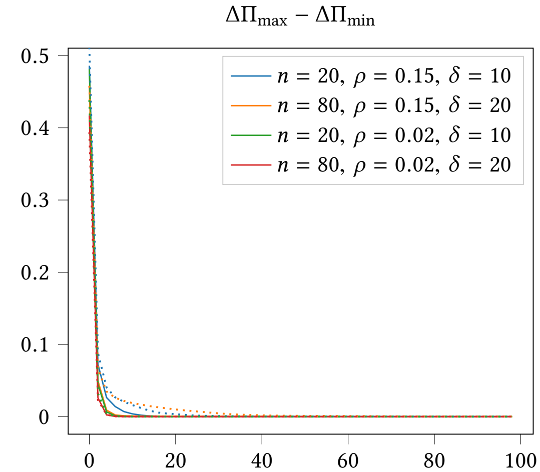


Figure 3: Directed and Undirected Graphs with $\varepsilon = \omega_0/5$**.**

cases the matrix product may not converge to an identical column matrix. Figure 3 shows the convergence of the matrix product for $\varepsilon = \omega_0/5$. Note that there does not seem to be much difference in the convergence for $\varepsilon = \omega_0/5$ or $\varepsilon = 0$.

At last, observe that the convergence in P_∞ seems to be much slower and noisy than for $\mathbf{y}_\infty$. This is consistent with the fact that $\mathbf{y}_\infty$ converges exponentially fast, while for P_∞ we do not have that guarantee, and thus may converge only as $t \longrightarrow \infty$. Observe that the convergence results are extremely similar for both directed and undirected graphs. This confirms the idea pointed out in Remark 6. Furthermore, The convergence to a P_∞ transition matrix seems to be heavily influenced by the evaporation rate.

6 DISCUSSION

The results in Section 4 show different kinds of convergence for a multi-agent swarm that follows a stigmergy based algorithm and what conditions the system needs to satisfy. Convergence of the probability transition matrix to P_∞ seems to happen only for $\varepsilon = 0$; To ensure convergence in agent distribution to a certain $\mathbf{y}_\infty$, the graph cannot become disconnected ($\varepsilon > 0$). Although, as seen in Figures 2b and 3, convergence in agent distribution seems to occur for most simulations even when $\varepsilon = 0$. Note as well that, as computed in Remark 7, the agent distribution convergence is exponentially fast and it can be seen in the simulation results. Allowing (or forcing) the graph to become disconnected in finite time, we would expect the agents to converge to a certain distribution only inside each sub-graph.

Additionally, from Corollary 3 and 4 we can now show that the convergence is maintained under communication constrains, therefore allowing agents to modify communication patterns based on convergence estimations. Therefore, now that we have guarantees that such a swarm will give rise to stationary behaviours, the main question that arises from these results is: How can we know more (and maybe control) the final distribution $\mathbf{y}_\infty$, and how do the swarm parameters affect this stationary distribution?. The authors consider this questions to be of big interest for robotic swarm design, and it is in fact the main line of work that the authors aim to pursue in the near future.

To address this problem, one could consider the swarm as the distribution $\mathbf{y}(t)$, considering an infinite number of agents, and study the resulting mean-field model. Representing the entire system as a mean field system (including the graph dynamics), would then open a venue to further study the target distributions $\mathbf{y}_\infty$.

7 PROOFS

Proof: Proposition 1. If there are no odd length cycles in $\mathcal{G}$, then we can split the graph in odd and even vertices. Starting from an odd vertex it is only possible to reach any other odd vertex in even times, and the converse. Let there now be one odd cycle C. Let i be a starting node and j any other vertex, with the shortest $i-j$ path being of even length l_{ij}. Then, $p^t_{ij} > 0$ if $t > 2k + l_{ij}$ $\forall k \in \mathbb{N}^+_0$. The only way of reaching j in odd time is by completing then the odd length cycle. Let l_{iv} be the minimum path length between i and any vertex $v \in C$, and let $l_v j$ be the minimum $v - j$ path length. Then, $p^t_{ij} > 0$ if $t > 2k + l_{iv} + l_{vj} + l_c$ $\forall k \in \mathbb{N}^+_0$. Since C is the only odd length cycle, $t_{\text{odd}} = 2k + l_{iv} + l_{vj} + l_c$ is an odd number. And particularly, if δ is the diameter of the graph,

$$t_{\text{odd}} \leq 2\delta + l_c.$$

□

Proof: Proposition 2. First of all, $\underline{W}_{ij} \in \{\varepsilon, 1\}$ for all edges $\{ij\}$ satisfying $W_{ij}(t) \neq 0$. Let $d^*_i := \max\{d_i : i \in \mathcal{V}'\}$. Then,

$$\alpha = \frac{\varepsilon}{1 + (d^*_i - 1)\varepsilon} \Rightarrow \underline{P}_{ji} > \alpha \quad \forall \underline{P}_{ji} > 0, \tag{9}$$

which satisfies the condition (2) of Assumption 1. For condition (1) in Assumption 1, see that the associated digraph to $P(t)$ is a connected planar graph. From Proposition 1, the matrix product

$$L^T(t_0, 2\delta{+}t_0{+}1) := \left[\underline{P}(t_0 + 2\delta + 1)\underline{P}(t_0 + 2\delta)...\underline{P}(t_0)\right]^T \tag{10}$$

has all entries $L^T_{kl}(t_0, 2\delta + t_0 + 1) > 0$ for any pair k, l and any t_0. This follows from connected graphs properties; each entry $L_{kl}(t_0, \delta + t_0)$ represents the probability of getting from vertex k to vertex l in δ steps starting from $t = t_0$. From (9) we make sure that the graph can never become disconnected, therefore $\underline{P}(t)$ is irreducible for all t. Furthermore, since no edges are being deleted for any t, the probability

$$\Pr\left[L^T(t_0, 2\delta + t_0 + 1) \in \mathcal{M}_2\right] = 1 \quad \forall t_0 > 0.$$

Hence, $\underline{P}^T(t)$ satisfies Assumption 1. □

Proof: Theorem 2. Let $\underline{P}(t)$ be constructed from Definition 11 with $\varepsilon > 0$ being a minimum weight at choice. From Proposition 2, we know that the sequence $\{\underline{P}^T(t)\}$ satisfies Assumption 1, and recalling Corollary 1, the left product

$$\lim_{k\to\infty} \prod_{t=0}^{k} \underline{P}(t) = \xi \mathbf{1}^T.$$

Then, the agent distribution as $t \to \infty$ is

$$\lim_{t\to\infty} \mathbf{y}(t) = \xi \mathbf{1}^T \mathbf{y}_0 = \xi,$$

since $\xi\mathbf{1}^T$ is a matrix of identical columns ξ and the vector $\mathbf{y}_0$ sums 1 over all its entries. The agent probability distribution converges *a.s.* to the vector ξ regardless of $\mathbf{y}_0$. □

Proof: Corollary 2. The proof follows identical steps to Theorem 2. Now we have $|\mathcal{A}|$ different sequences $\{P^a(t)\}$, depending on the movement of each agent. However, each sequence satisfies Assumption 1 (it can be easily checked by the logic in Proposition 2). Therefore, each agent converges to a distribution $\mathbf{y}^a(t) \xrightarrow{\text{a.s.}} \mathbf{y}^a_\infty$ as $t \to \infty$. □

Proof: Corollary 3. Proof is analogous to Theorem 2. In fact, to have $\mathbf{y}(t) \xrightarrow{\text{a.s.}} \mathbf{y}_\infty$ as $t \to \infty$ we do not need to impose γ to be independent from $M(t)$, but this requirement does affect a second corollary in the next section. □

Proof: Proposition 3. First see that if $W(t)$ is known, so is the transition probability matrix $P(t)$. Now recall that $P_{ji}(t)$ determines the probability of any agent moving from vertex i to vertex j at time t. Therefore, for any agent $a \in \mathcal{A}$,

$$P\{X_a(t+1) = j \,|\, X_a(t) = i\} = \begin{cases} 1, & i \in \mathcal{T}' \cup \mathcal{S}', \\ \frac{W_{ij}(t)}{w_i(t)} & \text{else.} \end{cases}$$

The weights in the graph are only updated after all agents have moved. Then, the choice of one agent at time t does not affect the choices of other agents at t. Denote $\mathcal{A}_i = \{a \in \mathcal{A} : X_a(t) = i\}$ and observe that $n_i(t) \equiv |\mathcal{A}_i|$. Then,

$$E\left[M_{ij}(t+1) \,|\, \mathcal{F}_t\right] = \sum_{a\in\mathcal{A}_i} P_{ji}(t) = P_{ji}(t) n_i(t).$$

□

Proof: Proposition 4. First, it is trivial from (3) that for any $i \in \mathcal{T}' \cup \mathcal{S}'$

$$\Delta P_{ji} = 0 \Rightarrow E\left[\Delta P_{ji} \,|\, \mathcal{F}_t\right] = 0 \quad \forall t > 0.$$

Consider now the rest of the edges ($i \notin \mathcal{T}' \cup \mathcal{S}'$). From (3) and substituting the weight dynamics in Definition 9:

$$P_{ji}(t{+}1) = \frac{W_{ij}(t+1)}{w_i(t+1)} = \frac{(1-\rho)W_{ij}(t) + \frac{\rho}{n}M_{ij}(t+1)}{w_i(t+1)}. \tag{11}$$

As pointed out in Remark 5

$$w_i(t+1) = (1-\rho)w_i(t) + \frac{\rho}{n}n_i(t). \tag{12}$$

Now we can compute the probability increment $\Delta P_{ji} = P_{ji}(t+1) - P_{ji}(t)$ from (11) as

$$\Delta P_{ji} = \frac{((1-\rho)W_{ij}(t) + M_{ij}(t+1)\frac{\rho}{n})w_i(t) - W_{ij}(t)w_i(t+1)}{w_i(t)w_i(t+1)} \tag{13}$$

and substituting (12) in the numerator in (13),

$$\Delta P_{ji} = \frac{\frac{\rho}{n}\left(M_{ij}(t+1)w_i(t) - W_{ij}(t)n_i(t)\right)}{w_i(t)w_i(t+1)}. \tag{14}$$

Observe that, by using the result in Proposition 3

$$\frac{W_{ij}(t)n_i(t)}{w_i(t)} = P_{ji}(t)n_i(t) = E[M_{ij}(t+1)\,|\mathcal{F}_t]. \quad (15)$$

Finally, substituting (15) in (14):

$$\Delta P_{ji} = \frac{\rho}{n}\frac{M_{ij}(t+1) - E[M_{ij}(t+1)\,|\mathcal{F}_t]}{w_i(t+1)}. \quad (16)$$

Let us now take the conditional expected value of (16). The denominator is fully determined by $\mathcal{F}_t$. Furthermore,

$$E[E[M_{ij}(t+1)\,|\mathcal{F}_t]\,|\,\mathcal{F}_t] = E[M_{ij}(t+1)\,|\mathcal{F}_t] \Rightarrow$$

$$\Rightarrow E[\Delta P_{ji}\,|\mathcal{F}_t] = \rho\frac{E[M_{ij}(t+1) - E[M_{ij}(t+1)|\mathcal{F}_t]\,|\mathcal{F}_t]}{nw_i(t+1)} = 0.$$

□

Proof: Theorem 4. Take the probability transition matrix increment $\Delta P_{ji}(t)$. See that it is a random variable that takes values $\Delta P_{ji}(t) \in [-1, 1]$ (therefore, $\sup_t E[\Delta P_{ji}(t)^+] < \infty$). Now, from Proposition 4

$$E[\Delta P_{ji}\,|\mathcal{F}_t] = 0 \Rightarrow E[P_{ji}(t+1) - P_{ji}(t)\,|\mathcal{F}_t] = 0.$$

See that $P_{ji}(t)$ is fully determined by the the information in σ-algebra $\mathcal{F}_t$. Then,

$$\begin{aligned} &E[P_{ji}(t+1) - P_{ji}(t)\,|\mathcal{F}_t] = E[P_{ji}(t+1)\,|\mathcal{F}_t] - P_{ji}(t) = \\ &=0 \iff E[P_{ji}(t+1)\,|\mathcal{F}_t] = P_{ji}(t). \end{aligned} \quad (17)$$

From Definition 4 it is clear that the entries of the probability transition matrix are all Martingales, and by Theorem 3 the matrix will converge to a P_∞ *a.s.* □

Proof: Corollary 4. Take eq. (14). If $\gamma \in \{0, 1\}$ is a random variable determining if weight is being added or not, we can write

$$\Delta P_{ji} = \frac{\frac{\rho}{n}\left(\sum_{k=1}^{M_{ij}(t+1)}\gamma_k - P_{ji}(t)\sum_{k=1}^{n_i(t)}\gamma_k\right)}{w_i(t+1)}, \quad (18)$$

with $w_i(t+1) = (1-\rho)w_i + \frac{\rho}{n}\sum_{k=1}^{n_i(t)}\gamma_k$. But if the variables $M(t)$ and γ are independent, $E[XY\,|\mathcal{F}_t] = E[X\,|\mathcal{F}_t]E[Y\,|\mathcal{F}_t]$. Furthermore, let $Z = \sum_{k=1}^{M_{ij}(t+1)}\gamma_k$, and observe that by the law of total expectation

$$\begin{aligned} &E\left[Z\,|\,\mathcal{F}_t\right] = E\left[E\left[Z\,|\,M_{ij}(t+1)\right]\,|\,\mathcal{F}_t\right] = \\ &= E\left[M_{ij}(t+1)p_\gamma\,|\,\mathcal{F}_t\right] = p_\gamma E\left[M_{ij}(t+1)\,|\,\mathcal{F}_t\right]. \end{aligned} \quad (19)$$

Then, taking the expected value of the numerator in (18):

$$\begin{aligned} &E\left[\sum_{k=1}^{M_{ij}(t+1)}\gamma_k - P_{ji}(t)\sum_{k=1}^{n_i(t)}\gamma_k\,\middle|\,\mathcal{F}_t\right] = \\ &=p_\gamma\left(E\left[M_{ij}(t+1)|\mathcal{F}_t\right] - E\left[M_{ij}(t+1)|\mathcal{F}_t\right]\right) = 0. \end{aligned} \quad (20)$$

Therefore, $P(t) \xrightarrow{\text{a.s.}} P_\infty$ as $t \to \infty$ regardless of γ. □

7.1 Construction of Expanded Graph

Definition 13. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph. Let $u \in \mathcal{V}$ and $\mathcal{E}_u \subset \mathcal{E}$ be the set of adjacent edges to u (in and out edges). We define the degree expansion of u with respect to a subset of the adjacent edges $\mathcal{E}_k \subset \mathcal{E}_u$ as a set of new vertices u_i and edges $u_i i$:*

$$C_\mathcal{V}(u, \mathcal{E}_k) := \{u_i : \{ui\} \in \mathcal{E}_k\},$$
$$C_\mathcal{E}(u, \mathcal{E}_k) := \{\{u_i i\} : \{ui\} \in \mathcal{E}_k\}$$

Note that $C_\mathcal{V}(u, \mathcal{E}_k)$ is a set of new vertices, all with degree 1, and $C_\mathcal{E}(u, \mathcal{E}_k)$ yields the edges connecting them to $\mathcal{E}_k$.

In Definition 13 we use undirected graph notation, such that $\{ui\} \equiv \{iu\}$. If $\mathcal{G}$ is directed, $C_\mathcal{E}(u, \mathcal{E}_k)$ and $C_\mathcal{V}(u, \mathcal{E}_k)$ are generated by computing in every case both (ui) and (iu).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W(t))$ be a connected, planar, weighted graph. Let $\mathcal{T} \subset \mathcal{V}, \mathcal{S} \subset \mathcal{V}$ be a target and starting set, with adjacent edge sets $\mathcal{E}_\mathcal{T}$, $\mathcal{E}_\mathcal{S}$ and frontier sets $\mathcal{T}_f = F(\mathcal{T} \to \mathcal{V} \setminus \mathcal{T})$, $\mathcal{S}_f = F(\mathcal{S} \to \mathcal{V} \setminus \mathcal{S})$. Let the sets of adjacent edges to the frontiers be

$$\mathcal{E}_{\mathcal{T},f} = \{\{uv\} : u \in \mathcal{T}_f,\, v \in \mathcal{V} \setminus \mathcal{T} \text{ or } v \in \mathcal{T}_f,\, u \in \mathcal{V} \setminus \mathcal{T}\},$$
$$\mathcal{E}_{\mathcal{S},f} = \{\{uv\} : u \in \mathcal{S}_f,\, v \in \mathcal{V} \setminus \mathcal{S} \text{ or } v \in \mathcal{S}_f,\, u \in \mathcal{V} \setminus \mathcal{S}\}.$$

The sets of expanded vertices and edges are

$$\mathcal{T}_x := \{\cup C_\mathcal{V}(u_T, \mathcal{E}_{\mathcal{T},f}) : u_T \in \mathcal{T}_f\},$$
$$\mathcal{S}_x := \{\cup C_\mathcal{V}(u_S, \mathcal{E}_{\mathcal{S},f}) : u_S \in \mathcal{S}_f\},$$

and

$$\mathcal{E}_{\mathcal{T}_x} := \{\cup C_\mathcal{E}(u_T, \mathcal{E}_{\mathcal{T},f}) : u_T \in \mathcal{T}_f\},$$
$$\mathcal{E}_{\mathcal{S}_x} := \{\cup C_\mathcal{E}(u_S, \mathcal{E}_{\mathcal{S},f}) : u_S \in \mathcal{S}_f\}.$$

Then, the expanded sets in $\mathcal{G}'$ are constructed as follows:

$$\mathcal{T}' := \mathcal{T}_x, \quad \mathcal{S}' := \mathcal{S}_x,$$
$$\mathcal{V}' := \mathcal{V} \cup (\mathcal{T}_x \cup \mathcal{S}_x) \setminus (\mathcal{T} \cup \mathcal{S}),$$
$$\mathcal{E}' := \mathcal{E} \cup (\mathcal{E}_{\mathcal{S}_x} \cup \mathcal{E}_{\mathcal{T}_x}) \setminus (\mathcal{E}_\mathcal{T} \cup \mathcal{E}_\mathcal{S}).$$

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