DISC Course: Modelling and Control of Hybrid Systems
Assignment HW2 (Lectures 3-5)

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Deadline: September, 23 2020

Preamble

This assignment deals with event-triggered control, an example of hybrid implementation of control laws. We revisit one of the most popular triggering techniques of the literature, which has been proposed by Paulo Tabuada in [5].

This assignment will give you the opportunity to confront yourself to three aspects seen in Lectures 3-5, namely the modelling, the stability analysis and the control of hybrid systems.

If you have any questions, feel free to send me an e-mail or to post them on the Discord server: `https://discord.gg/bQ8xt5`.

The reports have to be uploaded on DISC website by September, 23 2020.

Notation

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$, and $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$. The notation $\mathbb{R}^{n\times m}$ stands for the set of real matrices of n lines and m columns, with $n, m \in \mathbb{Z}_{>0}$. The notation (x, y) stands for $[x^\top,\ y^\top]^\top$, where $(x, y) \in \mathbb{R}^{n+m}$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $|x|$ and the distance of $x \in \mathbb{R}^n$ to a non-empty set $\mathcal{A} \subseteq \mathbb{R}^n$ is denoted by $|x|_{\mathcal{A}} := \inf\{|x-y| \ :\ y \in \mathcal{A}\}$. Let P be a real, square, and symmetric matrix, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are respectively the largest and the smallest eigenvalue of P. The notation $\mathbb{I}$ stands for the identity matrix, whose dimensions depend on the context.

Context

Nowadays, controllers are commonly implemented on digital devices and they often communicate with the plant sensors and/or actuators via a communication network. This network has a limited bandwidth and may be used by other tasks. In this context, it is essential to develop control strategies, which ensure the desired control law while using the digital channel only when needed.

Traditionally in practice, communications between the plant and the controller occur periodically, as in classical sampled-data control. While this approach is appealing due to its simplicity

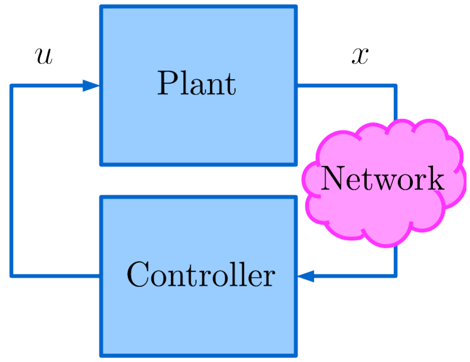

Figure 1: Control set-up

and the fact that a well-developed theory is available especially for linear systems, it may not be the best option when networks come into the control loop. Indeed, periodic sampling strategies generate transmissions irrespectively of the plant needs, and the sampling period is typically selected based on a worst case scenario. As a result, periodic strategies may lead to unnecessary frequent transmissions, and may thus overload the network. An alternative paradigm is event-triggered control. The idea is simple: communications between the plant and the controller only occur when this is needed according to the plant needs, just like we only exchange e-mails or talk to each other when we have something to say (in principle ;)). Consequently, transmissions are no longer defined using a clock but according to the plant state.

Event-triggered control can naturally be cast as a hybrid control problem for which a transmission, or communication, is modeled as a jump, and the triggering condition, i.e. the rule used to generate the transmission instants between the plant and the controller defines the flow and the jump sets, as we will see.

Questions

Consider the linear plant model

$$\dot{x} \quad = \quad Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$ with (A, B) stabilizable. The objective is to stabilize the origin of system (1), i.e. $x = 0$, using an event-triggered controller. We consider the scenario where the plant sensors transmit information to the controller over a network, and where the controller is directly connected to the plant actuators, as depicted on Figure 1.

We proceed by emulation, i.e. we first design a state-feedback law to stabilize the origin of system (1) ignoring the network. Hence,

$$u = Kx, \tag{2}$$

with $K \in \mathbb{R}^{n_u \times n_x}$ such that $A+BK$ is Hurwitz, i.e. the eigenvalues of $A+BK$ have strictly negative real parts. As a result, the system in closed-loop ignoring the network is $\dot{x} = (A+BK)x$ and $x = 0$ is uniformly globally asymptotically (exponentially, actually) stable.

Q1 (1pt). Can we always design K such that $A+BK$ is Hurwitz? Why?

We now take the network into account as in Figure 1. As a result, controller (2) has no longer access to $x(t)$ at any time instant $t \in \mathbb{R}_{\geq 0}$ but only to $x(t_k)$ for $t \in [t_k, t_{k+1})$ where t_k, $k \in \mathbb{Z}_{\geq 0}$, are the transmission instants to be defined.

Q2 (2pt). Introduce variable $\hat{x}$, the sampled version of x, which is generated using zero-order-hold devices as in Lecture 4, and derive a hybrid model of the closed-loop system of the form of $\mathcal{H}$ as defined in Lectures 3-4. Let the flow and the jump sets be undefined at the moment, i.e. simply use C and D; we will provide their definitions in the following. Do we need to introduce a clock variable as in Lecture 4 here?

The model developed in Q2 is fine, but it is more convenient to change the coordinates to proceed with the design of the flow and the jump sets and with the stability analysis. This is the purpose of the next question.

Q3 (1pt). Let $e := \hat{x} - x \in \mathbb{R}^{n_x}$ be the sampling-induced error, as it is the mismatch between the sampled value of x, $\hat{x}$, and its actual value x. Rewrite the hybrid model obtained in Q2 in the coordinates (x, e).

We are now going to design the triggering condition, i.e. the rule that we use to generate the transmission instants. We first establish a key property for this purpose.

Q4 (3pt). Since $A+BK$ is Hurwitz, there exists a unique symmetric, positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ such that[1]

$$(A+BK)^\top P + P(A+BK) = -\mathbb{I}.$$

Define $V : x \mapsto x^\top P x$ on $\mathbb{R}^{n_x}$. Prove that there exist $\underline{a}_V, \overline{a}_V, a, b > 0$ such that, for any

[1] See [3, Theorem 4.6].

$x, e \in \mathbb{R}^{n_x}$,

$$\begin{array}{l} \underline{a}_V |x|^2 \leq V(x) \leq \overline{a}_V |x|^2 \\ \langle \nabla V(x), (A+BK)x + BKe \rangle \leq -a|x|^2 + b|e|^2. \end{array} \tag{3}$$

Given (3), Paulo Tabuada has proposed to generate a transmission from the plant sensor to the controller whenever

$$b|e|^2 \geq \sigma a |x|^2, \tag{4}$$

where $\sigma \in (0,1)$ is a design parameter and e is defined in Q3. The underlying idea is the following. Since e is reset to 0 after a jump (in view of Q3), condition (4) enforces the next inequality for all $(t, j) \in \operatorname{dom}(x, e)$ (with $j \geq 1$ if $x(0,0) = 0$, otherwise j can be equal to 0) with (x, e) any solution to the system

$$b|e|^2 \leq \sigma a |x|^2, \tag{5}$$

which implies, in view of (3),

$$\begin{array}{rcl} \langle \nabla V(x), (A+BK)x + BKe \rangle & \leq & -a|x|^2 + b|e|^2 \\ & \leq & -a|x|^2 + \sigma a |x|^2 = -(1-\sigma)a|x|^2, \end{array} \tag{6}$$

which should preserve the original stability property of $x = 0$ as the right hand-side is strictly negative when $x \neq 0$. We are going to analyse this more carefully using the hybrid formalism presented in Lectures 3-5.

Q5 (3 pt). Finalize the hybrid model obtained in Q3 by providing the definition of the sets C and D in view of the triggering condition above. Make sure the obtained model satisfies the hybrid basic conditions, see Lecture 3. We call the obtained hybrid model[2] $\mathcal{H}_{\text{etc}}$ in the following.

The hybrid model is now complete. We can proceed with the stability analysis. We use one of the relaxed Lyapunov theorems of Lecture 4 for this purpose.

Q6 (4pt). Use V introduced in Q4 to check that the Lyapunov conditions of one of the relaxed Lyapunov theorems of Lecture 4 are verified. Make sure you carefully specify the attractor set $\mathcal{A}$. Note that only the properties related to V need to be established in this question, not those about the hybrid time domain of the solutions (these are addressed in the following).

Actually, we cannot use the exact same formulation of the relaxed Lyapunov theorem as we saw in Lecture 4 in view of the next question.

Q7 (2pt). Explain why there exist solutions to $\mathcal{H}_{\text{etc}}$ initialized at $(x(0,0), e(0,0)) = (0, e_0)$ with

[2] The index "etc" stands for event-triggered control.

$e_0 \in \mathbb{R}^{n_x}$, which violate the required property on the hybrid time domain as stated in the relaxed Lyapunov theorem.

It appears that this is not such an issue, as this defective behaviour can only happen in the attractor set. Indeed, it can be proved that that any solution to $\mathcal{H}_{\text{etc}}$ with initial condition of the form (x_0, e_0) with $x_0 \neq 0$ and $e_0 \in \mathbb{R}^{n_x}$ has a dwell-time[3]. In particular, there exists $\tau > 0$ such that for any solution (x, e) with $x(0,0) \neq 0$, for any $(t, j) \in \operatorname{dom}(x, e)$,

$$t + \tau \geq \tau j.$$

Hence, the desired property in the relaxed Lyapunov theorem on the hybrid time domains holds for such solutions, as explained in Lecture 4 (see slide 57). It is proved in [4] that the issue with the initialization in $\{0\} \times \mathbb{R}^{n_x}$ is actually not an problem: we can still conclude that $\mathcal{A}$ obtained in Q6 is uniformly globally pre-asymptotically stable. We can go even further.

Q8 (3pt). Prove that maximal solutions to $\mathcal{H}_{\text{etc}}$ are complete using Lecture 3. As a result, set $\mathcal{A}$ is uniformly globally asymptotically stable for $\mathcal{H}_{\text{etc}}$.

We finally test the designed event-triggered controller on a numerical example.

Q9 (6pt). Consider the double integrator case, i.e. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in (1). We design controller (2) with $K = (-2 \ \ -2)$; you can verify that $A + BK$ is indeed Hurwitz. It can be proved that (3) holds with $a = \frac{1}{2}$ and $b = 13$. Simulate the event-triggered controlled system using your favourite software[4] for different values of $\sigma \in (0,1)$. Numerically study the impact of σ on the evolution of the x-component of the solutions as well as on the number of jumps/transmissions for a given amount of continuous time.

Conclusion

If you like to know more about event-triggered control, here are some survey style articles: [1,2].

Also, any feedback or suggestions from your side in your report or by e-mail will be appreciated. Thanks in advance and good luck!

[3] See [4,5] for more details.

[4] If you opt for Matlab-Simulink, I recommend the hybrid toolbox developed by Ricardo Sanfelice et al, which is freely available here: `https://www.mathworks.com/matlabcentral/fileexchange/41372-hybrid-equations-toolbox-v2-04`.

References

[1] W.P.M.H. Heemels, K.H. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *IEEE Conference on Decision and Control, Maui, U.S.A.*, pages 3270–3285, 2012.

[2] W.P.M.H. Heemels, R. Postoyan, M.C.F. Donkers, A.R Teel, A. Anta, P. Tabuada, and D. Nešić. *Periodic Event-triggered Control*, chapter 6, pages 105–119. CRC Press / Taylor & Francis, 2015.

[3] H.K. Khalil. *Nonlinear systems*. Prentice-Hall, Englewood Cliffs, New Jersey, U.S.A., 3rd edition, 2002.

[4] R. Postoyan, P. Tabuada, D. Nešić, and A. Anta. A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4):982–996, 2015.

[5] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Trans. on Automatic Control*, 52(9):1680–1685, 2007.