Lecture 4: Stability and Robustness of Hybrid Systems

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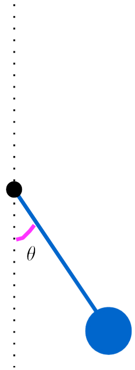
Stability, an intuitive treatment: equilibria and stability

Equilibrium points: once there, we do not move!



2 equilibria: upward and downward positions

What do we want to call a stable/unstable equilibrium?

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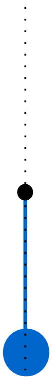

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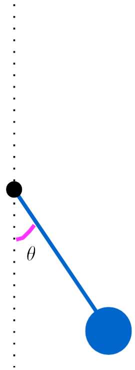

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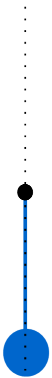

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Stability, an intuitive treatment: main ideas

An equilibrium is **stable** if, when we start close to it, we remain close to it for all future times (and we can keep moving!).

$\rightarrow$ downward position of the pendulum

An equilibrium is **unstable** if it is not stable.

$\rightarrow$ upward position of the pendulum

An equilibrium is **locally asymptotically stable** if

- it is stable,
- solutions initialized nearby converge asymptotically to it: we talk of **attractivity**.

$\rightarrow$ downward position of the pendulum when taking friction into account

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Stability, an intuitive treatment: remarks

Important remarks:

- We say that a (equilibrium) point is (locally, globally, asymptotically) stable for a system and not that the system is stable.
- Asymptotic stability is not the same as asking solutions to converge asymptotically to the considered equilibrium: we also need stability.

  Vinograd counterexample:

$$\begin{aligned} \dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{r^2(1 + r^4)} \\ \dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{r^2(1 + r^4)}, \end{aligned}$$

  où $r^2 = x_1^2 + x_2^2$, cf. animation.

  For linear time-invariant systems, asymptotic convergence is equivalent to asymptotic stability.
- Asymptotic stability is a fundamental notion in control, which (should) ensure nominal robustness properties.

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Often in control, we study the stability of the origin, i.e. $x = 0$.

We can always translate the stability of an equilibrium $x = x^\star \neq 0$ to the stability of the origin.

Consider the nonlinear continuous-time

$$\dot{x} = f(x)$$

and suppose $f(x^\star) = 0$, i.e. $x^\star$ is an equilibrium point of the system.

Define $z = x - x^\star$. Then

$$\dot{z} = \dot{x} - \dot{x}^\star = \dot{x} = f(x) = f(z + x^\star) =: g(z),$$

and we have $g(0) = f(x^\star) = 0$: $z = 0$ is the equilibrium to the new system.

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Stability, an intuitive treatment: towards set stability

After all, $x = 0$ is nothing but a special set, namely $\{0\}$.

We should therefore be able to extend the notion of stability to more general sets.

What is the natural notion of equilibrium for non-singleton sets?
$\rightarrow$ **invariance**, i.e. when the system is initialized in the set, it remains there for all future times.

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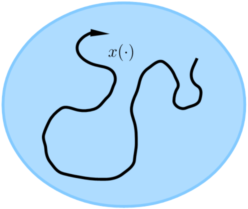

Stability, an intuitive treatment: set stability

"Same as before"

A **set** is **stable** if, when we start close to it, we remain close to it for all future times.

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Stability, an intuitive treatment: distance to a set

What do we mean by "initialized closed to the set"?

When studying the origin, we usually take $|x|$.

When studying a set $\mathcal{A} \subseteq \mathbb{R}^n$, we take the **distance to the set**

$$|x|_{\mathcal{A}} := \inf \{|x - y| \; : \; y \in \mathcal{A}\}$$

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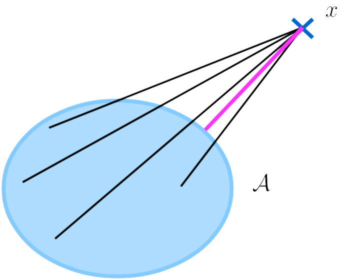


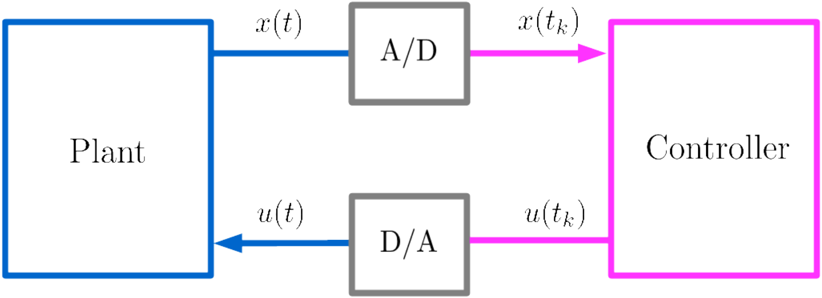
Stability, an intuitive treatment: why?

Yes, in particular when dealing with hybrid systems.

Examples:

- Sampled-data control
- Switched systems
- Time-varying systems

Stability, an intuitive treatment: sampled-data control



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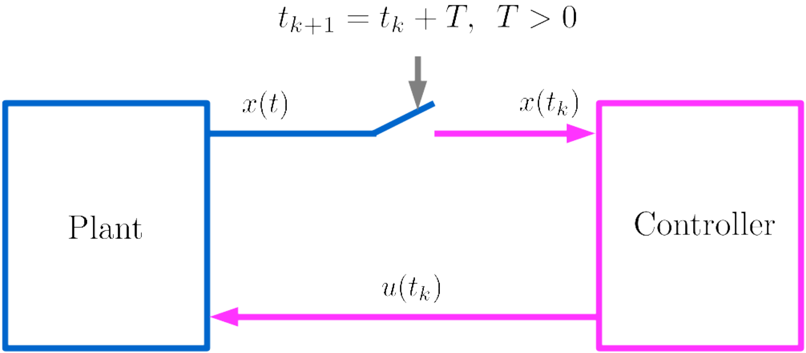

Stability, an intuitive treatment: sampled-data control

Consider the plant model

$$\dot{x} = Ax + Bu$$

and the controller

$$u = Kx,$$

which is implemented using a zero-order-hold device so that

$$u(t) = Kx(t_k), \qquad \forall t \in [t_k, t_{k+1}).$$

The sampling instants t_k, $k \in \mathbb{Z}_{\geq 0}$, are such that

$$t_{k+1} = t_k + T,$$

where $T > 0$ is the sampling period.

The system in closed-loop is given by

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Instead of working with $x(t_k)$, we introduce a new variable $\hat{x}$, which is such that

$$\dot{\hat{x}} = 0, \ \forall t \in [t_k, t_{k+1}), \qquad \hat{x}(t_k^+) = x(t_k)$$

Hence

$$\hat{x}(t) = x(t_k) \quad \forall t \in [t_k, t_{k+1}) \qquad (\text{for } k \geq 1)$$

Let us get rid of "$[t_k, t_{k+1})$". We introduce for this purpose the clock variable $\tau \in \mathbb{R}_{\geq 0}$,

$$\dot{\tau} = 1 \ \forall t \in [t_k, t_{k+1}), \qquad \tau^+ = 0.$$

When do we jump, i.e. sample? $\rightarrow$ when $\tau = T$

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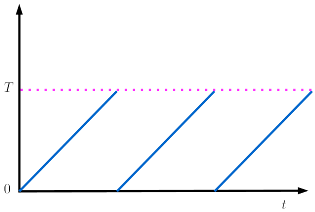

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We thus have the next hybrid system

$$\left.\begin{array}{rcl} \dot{x} & = & Ax + BK\hat{x} \\ \dot{\hat{x}} & = & 0 \\ \dot{\tau} & = & 1 \end{array}\right\} \quad \tau \in [0, T]$$

$$\left.\begin{array}{rcl} x^+ & = & x \\ \hat{x}^+ & = & x \\ \tau^+ & = & 0 \end{array}\right\} \quad \tau = T$$

Suppose our original goal was to stabilize $x = 0$, now it becomes to stabilize

$$\mathcal{A} = \{0\} \times \{0\} \times [0, T]$$

No hope to reduce the problem to the analysis of the stability of the origin $x = 0$, $\hat{x} = 0$ and $\tau = 0$.

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Stability, an intuitive treatment: switched systems

Consider the system

$$\dot{x} = f_\sigma(x),$$

where $\sigma \in \{1, ..., N\}$ is the switching signal, $N \in \mathbb{Z}_{>0}$.

Suppose switches occur according to time (and not state, but it is not important for our discussion).

We thus have a (general) clock

$$\dot{\tau} \in H(\tau), \;\; \tau^+ = 0$$

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We saw how to convert a time-varying system into an autonomous one

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix} \in \begin{pmatrix} F(t,x) \\ 1 \end{pmatrix} = \widetilde{F}(z)$$

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$$\mathcal{A} = \{0\} \times \mathbb{R}_{\geq 0}$$

Stability, an intuitive treatment: a final remark

It is very important to carefully model the system under consideration with all its state variables, and to carefully define the set, whose stability is studied.

Stability, an intuitive treatment: outline

What's next?

- Mathematical formulation of set stability
- Are these notions robust?
- How to check stability? $\rightarrow$ Lyapunov theorems and an invariance result

Overview

1. Stability, an intuitive treatment
2. Definition
3. Main Lyapunov theorem
4. Relaxed Lyapunov theorems and an invariance result
5. Discussions
6. Summary

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Definition

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$, if:

- it is continuous,
- $\alpha(0) = 0$,
- it is strictly increasing,
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Examples: for $s \in \mathbb{R}_{\geq 0}$,

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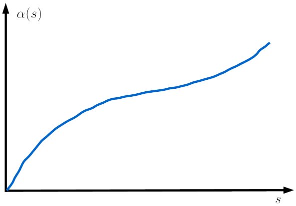

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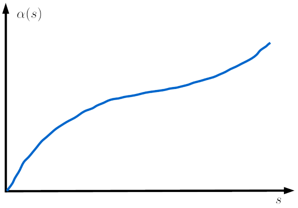

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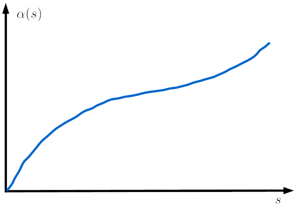

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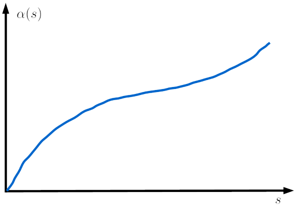

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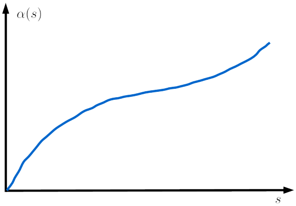

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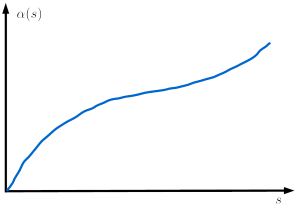

Definition: uniform global stability (UGS)

Recall

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \tag{$\mathcal{H}$}$$

Definition

Consider system $\mathcal{H}$. The closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be:

- **uniformly globally stable** if there exists $\alpha \in \mathcal{K}_\infty$ such that for any solution ϕ

$$|\phi(t,j)|_{\mathcal{A}} \leq \alpha\left(|\phi(0,0)|_{\mathcal{A}}\right),$$

for all $(t,j) \in \operatorname{dom}\phi$.

"If we start close, we remain close:" if $|\phi(0,0)|_{\mathcal{A}} \leq \varepsilon$ (small), then $|\phi(t,j)|_{\mathcal{A}} \leq \alpha(\varepsilon)$ (small) for all $(t,j) \in \operatorname{dom}\phi$.

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Definition: uniform global pre-asymptotic stability (UGpAS)

Definition

- **uniformly globally pre-attractive** if

$$\forall \varepsilon, r > 0 \; \exists T > 0 \; \forall \text{ solution } \phi$$
$$|\phi(0,0)|_{\mathcal{A}} \leq r \;\; \Rightarrow \;\; |\phi(t,j)|_{\mathcal{A}} \leq \varepsilon \text{ for } (t,j) \in \operatorname{dom} \phi \text{ and } t + j \geq T.$$

- **uniformly globally pre-asymptotically stable** if it is both uniformly globally stable and uniformly globally pre-attractive

- We remove the prefix "-pre" when maximal solutions are complete.

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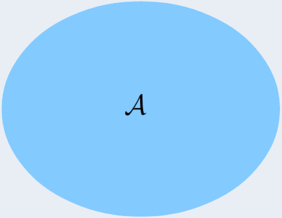

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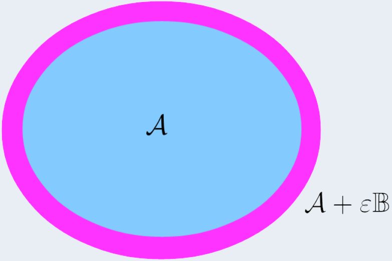

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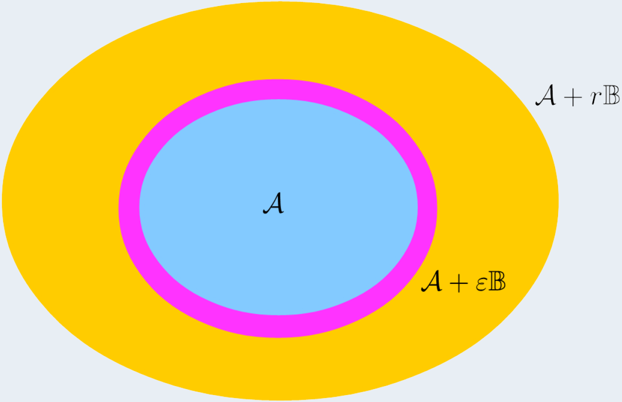

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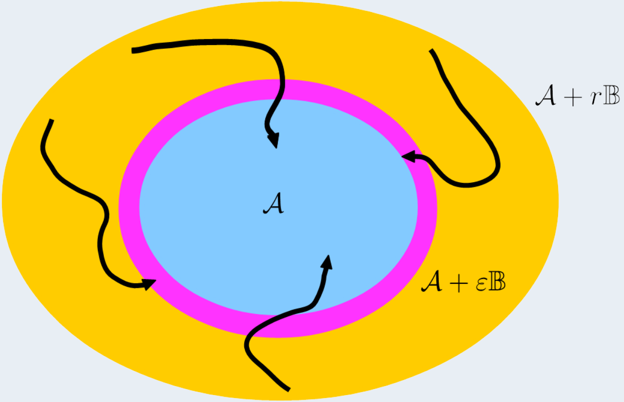

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Definition: questions

Why pre-?

→ Stability says nothing about the hybrid time domains of the solutions, and thus about completeness of maximal solutions.

Take

$$\left.\begin{array}{rcl}\dot{x}_1 & = & x_1^2 \\ \dot{x}_2 & = & -x_2,\end{array}\right\} (x_1, x_2) \in \mathbb{R} \times \mathbb{R}$$

and $D = \emptyset$ and let $\mathcal{A} = \{x = (x_1, x_2) \,:\, x_2 = 0\}$.

For any solution x and $(t, 0) \in \operatorname{dom} x$, $x_2(t, 0) = e^{-t} x_2(0, 0)$, so

$$|x(t,0)|_{\mathcal{A}} = |x_2(t,0)| \leq |x_2(0,0)| = \alpha(|x_2(0,0)|) = \alpha(|x(0,0)|_{\mathcal{A}})$$

with $\alpha(s) = s$ for any $s \geq 0$ (uniform global stability).

We see that x_2 should converge to 0 as time grows.

For any $x_1(0,0) > 0$ and $x_2(0,0)$, solutions are only defined on $\left[0, \frac{1}{x_1(0,0)}\right) \times \{0\}$

However, we have that $\mathcal{A}$ is uniformly globally pre-attractive as the property holds (vacuously for $T > \frac{1}{x_1(0,0)}$ when $x_1(0,0) > 0$). This is due to the fact that $\mathcal{A}$ is not

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and $D = \emptyset$ and consider the **compact** attractor

$$\mathcal{A} = \{0\} \times [0,1]$$

Consider a solution x, which flows. Hence there exists $t \geq 0$ such that $(t,0) \in \operatorname{dom} \phi$. We have

$$x_1(t,0) = e^t x_1(0,0)$$

consequently,

$$|x(t,0)|_{\mathcal{A}} = |x_1(t,0)| = e^t |x_1(0,0)|$$

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Definition: questions

Consequently, for any solution x,

- $\sup_t \operatorname{dom} x \leq 1$
- $\sup_j \operatorname{dom} x \leq 0$.

We derive that the uniform global pre-attractivity property holds by taking $T > 1$.

Concerning uniform global stability, we have that, for any solution x and all $(t, j) \in \operatorname{dom} x$, necessarily $j = 0$ and

$$|x(t,0)|_{\mathcal{A}} = |x_1(t,0)| \leq e^1 |x_1(0,0)| = \alpha(|x(0,0)|_{\mathcal{A}}),$$

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Definition: questions

More counter-intuitive examples are given in Chapter 3.1 of [Goebel et al., 2012].

How to guarantee that maximal solutions are complete?
→ we saw conditions for that in the previous lecture.

Again, keep in mind that stability and properties of the solution hybrid time domains (and so completeness) are two different things.

Not the case where studying the stability of the origin for differential/difference equations → stability ensures complete maximal solutions.

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Definition: $\mathcal{KL}$-characterization

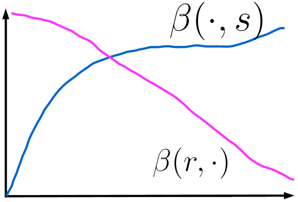
Definition

A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$, $\beta \in \mathcal{KL}$, if it is:

- nondecreasing in its first argument,
- nonincreasing in its second argument,
- $\beta(r, s) \to 0$ as $r \to 0$, for any $s \in \mathbb{R}_{\geq 0}$,
- $\beta(r, s) \to 0$ as $s \to \infty$, for any $r \in \mathbb{R}_{\geq 0}$.

Examples: for any $r, s \in \mathbb{R}_{\geq 0}$,

- $\beta(r, s) = re^{-s}$ ✓,
- $\beta(r, s) = \lambda_1 r^2 e^{-\lambda_2 s}$, for some $\lambda_1, \lambda_2 > 0$ ✓,
- $\beta(r, s) = r\frac{1}{1+s}$ ✓.



Definition: $\mathcal{KL}$-characterization

Hybrid system

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \tag{$\mathcal{H}$}$$

Theorem

Let closed set $\mathcal{A} \subseteq \mathbb{R}^n$ and consider system $\mathcal{H}$. The following statements are equivalent:

- $\mathcal{A}$ is UGpAS.
- There exists $\beta \in \mathcal{KL}$ such that for any solution ϕ,

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta\left(|\phi(0,0)|_{\mathcal{A}}, t+j\right), \qquad \forall (t,j) \in \operatorname{dom}\phi.$$

Definition: is this notion robust?

It would not be natural to talk of stability if it would not come with some robustness properties.

The "weakest" notion of robustness is the following.

Consider the perturbed system, as in the previous chapter, where $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (continuous typically)

$$\begin{cases} \dot{x} & \in & F_\rho(x) & x \in C_\rho \\ x^+ & \in & G_\rho(x) & x \in D_\rho, \end{cases} \qquad (\mathcal{H}_\rho)$$

where

$$\begin{array}{rcl} C_\rho & = & \{x : (x + \rho(x)\mathbb{B}) \cap C \neq \emptyset\} \text{ "} = C \text{ inflated by something of the order of } \rho(x)\text{"} \\ D_\rho & = & \{x : (x + \rho(x)\mathbb{B}) \cap D \neq \emptyset\} \text{ "} = D \text{ inflated by something of the order of } \rho(x)\text{"} \\ F_\rho(x) & = & \overline{\text{con}} F\left((x + \rho(x)\mathbb{B}) \cap C\right) + \rho(x)\mathbb{B} \;\; \forall x \in \mathbb{R}^n, \text{ "} = f(x + \rho(x)) + \rho(x)\text{"} \\ G_\rho(x) & = & \{v \in \mathbb{R}^n : v \in g + \rho(g)\mathbb{B}, \, g \in G\left((x + \rho(x)\mathbb{B}) \cap D\right)\} \;\; \forall x \in \mathbb{R}^n \\ & = & \text{"}g(x + \rho(x)) + \rho(x)\text{"}. \end{array}$$

and $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$

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$$\begin{array}{rcl} C_\rho & = & \{x : (x + \rho(x)\mathbb{B}) \cap C \neq \emptyset\} \text{ "} = C \text{ inflated by something of the order of } \rho(x)\text{"} \\ D_\rho & = & \{x : (x + \rho(x)\mathbb{B}) \cap D \neq \emptyset\} \text{ "} = D \text{ inflated by something of the order of } \rho(x)\text{"} \end{array}$$

$$F_\rho(x) = \overline{\mathrm{con}} F\left((x + \rho(x)\mathbb{B}) \cap C\right) + \rho(x)\mathbb{B} \;\; \forall x \in \mathbb{R}^n, \text{ "} = f(x + \rho(x)) + \rho(x)\text{"}$$

$$\begin{array}{rcl} G_\rho(x) & = & \{v \in \mathbb{R}^n : v \in g + \rho(g)\mathbb{B}, \, g \in G\left((x + \rho(x)\mathbb{B}) \cap D\right)\} \;\; \forall x \in \mathbb{R}^n \\ & = & \text{"}g(x + \rho(x)) + \rho(x)\text{"}. \end{array}$$

and $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$

Definition: is this notion robust?

It would not be natural to talk of stability if it would not come with some robustness properties.

The "weakest" notion of robustness is the following.

Consider the perturbed system, as in the previous chapter, where $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (continuous typically)

$$\begin{cases} \dot{x} & \in & F_\rho(x) & & x \in C_\rho \\ x^+ & \in & G_\rho(x) & & x \in D_\rho, \end{cases} \qquad (\mathcal{H}_\rho)$$

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$$= \text{"}g(x+\rho(x)) + \rho(x)\text{"}.$$

and $\mathbb{B}$ is the unit ball of $\mathbb{R}^n$

Definition: robustly UGpAS

Definition

We say that a **compact** set $\mathcal{A} \subset \mathbb{R}^n$ is **robustly UGpAS** if there exists ρ:

- continuous
- positive on $\Big(C \cup D \cup G(D)\Big) \backslash \mathcal{A}$

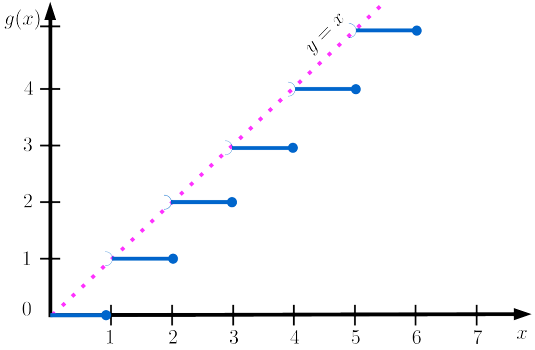
such that $\mathcal{A}$ is UGpAS for system $\mathcal{H}_\rho$.

Definition: non-robust UGpAS example

Counter-example

$$x^+ \quad = \quad g(x) \quad x \in [0, \infty)$$

and $C = \emptyset$.



$\mathcal{A} = \{0\}$ is UGpAS but this property has zero robustness
The map is not outer-semicontinuous $\rightarrow$ one of the basic conditions is not satisfied
When we regularize the jump map, $\mathcal{A} = \{0\}$ is no longer UGpAS.

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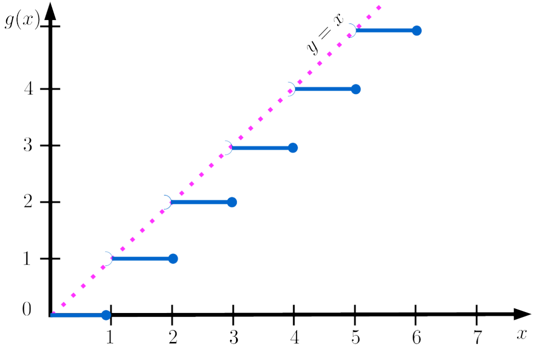

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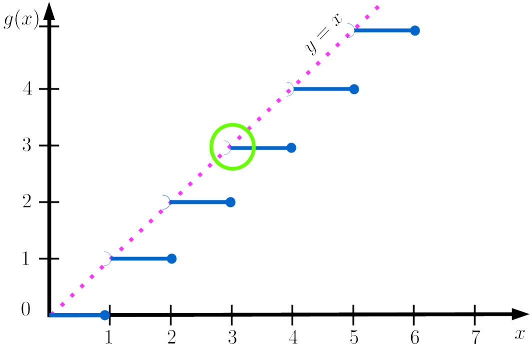

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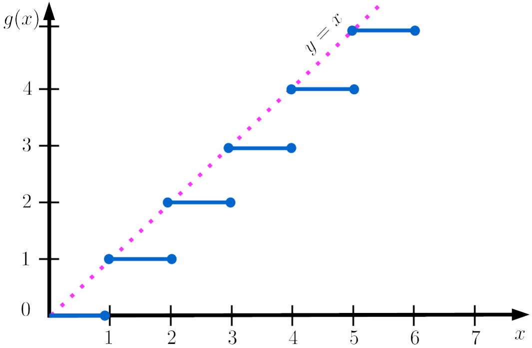

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Definition: conditions for robust UGpAS

Theorem

If $\mathcal{A}$ is **compact**, UGpAS for system $\mathcal{H}$, which satisfies the hybrid basic conditions, then it is **robustly UGpAS**.

Definition: how to prove stability?

OK, but how can we check that a given set satisfies stability properties?

$\rightarrow$ need to compute the solution $\rightarrow$ very difficult in general, if not impossible

Even for linear time-invariant systems, we did not compute the solutions to assess whether the origin is stable

$$\dot{x} = Ax$$

$\rightarrow$ study the eigenvalues of A.

Hybrid system:

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D.$$

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Overview

1. Stability, an intuitive treatment
2. Definition
3. Main Lyapunov theorem
4. Relaxed Lyapunov theorems and an invariance result
5. Discussions
6. Summary

Main Lyapunov theorem: outline of this section

- Differential equations (continuous-time)
- Differential inclusions (continuous-time)
- Difference equations (discrete-time)
- Difference inclusions (discrete-time)
- Hybrid systems

Main Lyapunov theorem: differential equations

Consider

$$\dot{x} = f(x), \qquad \text{(CT)}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be closed.

Theorem

If there exist:

- $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous differentiable,
- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$,

such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$$

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_\mathcal{A}),$$

then the set $\mathcal{A}$ is UGpAS for system CT.

Main Lyapunov theorem: differential equations, comments

Key role: V the so-called **Lyapunov function**.

For any $x \in \mathbb{R}^n$, $V(x)$ is a nonnegative scalar.

First property: for all $x \in \mathbb{R}^n$,

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Magenta part implies that:

- V is positive for any $x \notin \mathcal{A}$, as in this case, $|x|_{\mathcal{A}} \neq 0$ and so $0 < \alpha_1(|x|_{\mathcal{A}}) \leq V(x)$.
- V is radially unbounded with respect to $\mathcal{A}$. Indeed, as $|x|_{\mathcal{A}} \to \infty$, $\alpha_1(|x|_{\mathcal{A}}) \to \infty$ and so does $V(x)$.

Blue part: when $x \in \mathcal{A}$, $|x|_{\mathcal{A}} = 0$ and thus $\alpha_1(|x|_{\mathcal{A}}) = \alpha_2(|x|_{\mathcal{A}}) = 0$. Thus, $V(x) = 0$.

"V is positive definite and radially unbounded with respect to $\mathcal{A}$"

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Main Lyapunov theorem: differential equations, comments

Second property: for any $x \in \mathbb{R}^n$,

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_{\mathcal{A}}),$$

Why $\langle \nabla V(x), f(x) \rangle$?

This essentially corresponds to $\dot{V}(x(t))$, indeed by the chain rule

$$\begin{aligned} \dot{V}(x(t)) &= \frac{d}{dt} V(x(t)) = \frac{d}{dx} V(x(t)) \frac{d}{dt} x(t) \\ &= \frac{d}{dx} V(x(t)) f(x(t)) = \langle \nabla V(x(t)), f(x(t)) \rangle \end{aligned}$$

Why not to write $\dot{V}(x(t))$ then?

- Because x is a solution in $\dot{V}(x(t))$, and so a function of the time, which may not be defined for all times as we saw.
- On the other hand, in $\langle \nabla V(x), f(x) \rangle$, x is a vector of $\mathbb{R}^n$ and we do not have to worry about the existence of solutions. Also, we clearly see which "system" (vector field here) we are considering.

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Main Lyapunov theorem: differential equations, comments

Recall: for any $x \in \mathbb{R}^n$,

$$\langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_{\mathcal{A}}),$$

We ask $\langle \nabla V(x), f(x) \rangle$ to strictly decrease as long as the state is not in $\mathcal{A}$.

We do not need to compute solution to check the above condition.

Uniform global stability? just take $\rho = 0$.

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Main Lyapunov theorem: differential equations, example

Consider

$$\dot{x} = -x^3.$$

Let $V(x) = x^2$ for any $x \in \mathbb{R}$.

We take $\alpha_1(s) = \alpha_2(s) = s^2$ for any $s \geq 0$ and we have that, for any $x \in \mathbb{R}$,

$$\alpha_1(|x|) = V(x) = \alpha_2(|x|).$$

On the other hand, for $x \in \mathbb{R}$, $\nabla V(x) = 2x$, so

$$\langle \nabla V(x), f(x) \rangle = \left\langle 2x, -x^3 \right\rangle = -2x^4 = -\rho(|x|)$$

with $\rho(s) = 2s^4$ for any $s \geq 0$. We derive that $x = 0$ is UG(p)AS.

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Consider

$$\dot{x} \in F(x), \tag{CT-incl}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be closed.

Theorem

If there exist:

- $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous differentiable,
- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$,

such that, for all $x \in \mathbb{R}^n$ and any $f \in F(x)$,

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$
$$\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}),$$

then the set $\mathcal{A}$ is UGpAS for system CT-incl.

Main Lyapunov theorem: difference equations

Consider

$$x^+ = g(x), \tag{DT}$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be closed.

Theorem

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- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$,

such that, for all $x \in \mathbb{R}^n$,

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$$
$$V(g(x)) - V(x) \leq -\rho(|x|_\mathcal{A}),$$

then the set $\mathcal{A}$ is UGpAS for system DT.

Instead of writing V^+ or $V(x^+) \leq -\rho(|x|_\mathcal{A})$, we use $V(g(x)) \leq -\rho(|x|_\mathcal{A})$ for similar reasons as before.

Main Lyapunov theorem: difference inclusions

Consider

$$x^{+} \in G(x), \qquad \text{(DT-incl)}$$

where $G : R^n \rightrightarrows \mathbb{R}^n$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be closed.

Theorem

If there exist:

- $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous differentiable,
- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$,

such that, for all $x \in \mathbb{R}^n$, for any $g \in G(x)$,

$$\begin{array}{c} \alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \\ V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}), \end{array}$$

then the set $\mathcal{A}$ is UGpAS for system DT-incl.

Main Lyapunov theorem: hybrid inclusions

Consider

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \qquad (\mathcal{H})$$

Theorem

If there exist:

- $V : \text{dom } V \to \mathbb{R}_{\geq 0}$,
- $C \cup D \cup G(D) \subset \text{dom } V$,
- V is continuous differentiable on a open set containing C,
- $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,
- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$,

such that

$$\begin{aligned} &\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) && \forall x \in C \cup D \cup G(D) \\ &\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) && \forall x \in C,\ f \in F(x) \\ &V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) && \forall x \in D,\ g \in G(x), \end{aligned}$$

then the set $\mathcal{A}$ is UGpAS for system $\mathcal{H}$.

Main Lyapunov theorem: main result

Consider

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D, \qquad (\mathcal{H})$$

Recall

$$\begin{array}{ll} \langle \nabla V(x), f\rangle \leq -\rho(|x|_{\mathcal{A}}) & \forall x \in C,\ f \in F(x) \\ V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) & \forall x \in D,\ g \in G(x), \end{array}$$

Why the same ρ on flows and at jumps?

$\rightarrow$ if a ρ_c for flow and a ρ_d at jumps, define

$$\rho = \min(\rho_c, \rho_d).$$

Main Lyapunov theorem: example, the bouncing ball

Consider

$$\begin{cases} \dot{x} \in \begin{cases} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x \neq 0 \\ \begin{pmatrix} 0 \\ [-\gamma, 0] \end{pmatrix} & x = 0 \end{cases} & x_1 \geq 0 \\ x^+ = \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} & x_1 = 0 \text{ and } x_2 \leq 0. \end{cases}$$

Let $x = (x_1, x_2) \in C \cup D \cup G(D)$, $\mathcal{A} = \{(0,0)\}$, and

$$V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1.$$

We have that

$$\alpha_1(|x|) \leq V_1(x) \leq \alpha_2(|x|),$$

with $\alpha_1(s) = \min\left\{\frac{1}{2}(s/\sqrt{2})^2, \frac{\gamma}{\sqrt{2}}s\right\}$ and $\alpha_2(s) = \dfrac{1}{2}s^2 + s$ for any $s \geq 0$.

Main Lyapunov theorem: example, the bouncing ball

Consider

$$\left\{ \begin{array}{lll} \dot{x} & \in & \left\{ \begin{array}{ll} \begin{pmatrix} x_2 \\ -\gamma \end{pmatrix} & x \neq 0 \\ \begin{pmatrix} 0 \\ [-\gamma, 0] \end{pmatrix} & x = 0 \end{array} \right. \quad x_1 \geq 0 \\ x^+ & = & \begin{pmatrix} x_1 \\ -\lambda x_2 \end{pmatrix} \quad x_1 = 0 \text{ and } x_2 \leq 0. \end{array} \right.$$

Let $x = (x_1, x_2) \in C \cup D \cup G(D)$, $\mathcal{A} = \{(0,0)\}$, and

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Main Lyapunov theorem: example, the bouncing ball

Consider

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Let $x = (x_1, x_2) \in C \cup D \cup G(D)$, $\mathcal{A} = \{(0,0)\}$, and

$$V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1.$$

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with $\alpha_1(s) = \min\left\{\frac{1}{2}(s/\sqrt{2})^2, \frac{\gamma}{\sqrt{2}}s\right\}$ and $\alpha_2(s) = \dfrac{1}{2}s^2 + s$ for any $s \geq 0$.

Main Lyapunov theorem: example, the bouncing ball

Recall

$$V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1.$$

Let $x \in C$ and $f \in F(x)$,

$$\langle \nabla V_1(x), f\rangle = \langle (\gamma, x_2), f\rangle$$

when $f = (x_2, -\gamma)$,

$$\langle (\gamma, x_2), (x_2, -\gamma)\rangle = \gamma x_2 - \gamma x_2 = 0$$

when $f \in (0, [-\gamma, 0])$, $f = (0, a)$ with $a \in [-\gamma, 0]$ and this can only happen when $x = 0$, hence

$$\langle (\gamma, 0), (0, a)\rangle = 0.$$

We do not have the expected property, i.e. no strict decrease on flows!

Let $x \in D$, (recall that $x_1^+ = x_1 = 0$ and $x_2^+ = -\lambda x_2$)

$$\begin{aligned} V_1(g(x)) - V_1(x) &= \tfrac{1}{2}(x_2^+)^2 + \gamma x_1^+ - \tfrac{1}{2}x_2^2 - \gamma x_1 \\ &= \tfrac{1}{2}(-\lambda x_2)^2 - \tfrac{1}{2}x_2^2 \\ &= -\tfrac{1}{2}(1-\lambda^2)x_2^2 \\ &= -\tfrac{1}{2}(1-\lambda^2)(x_1^2 + x_2^2) = -\rho(|x|). \end{aligned}$$

Main Lyapunov theorem: example, the bouncing ball

Recall

$$V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1.$$

Let $x \in C$ and $f \in F(x)$,

$$\langle \nabla V_1(x), f\rangle \quad = \quad \langle (\gamma, x_2), f\rangle$$

when $f = (x_2, -\gamma)$,

$$\langle (\gamma, x_2), (x_2, -\gamma)\rangle \quad = \quad \gamma x_2 - \gamma x_2 = 0$$

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$$\langle (\gamma, 0), (0, a)\rangle \quad = \quad 0.$$

We do not have the expected property, i.e. no strict decrease on flows!

Let $x \in D$, (recall that $x_1^+ = x_1 = 0$ and $x_2^+ = -\lambda x_2$)

$$\begin{aligned} V_1(g(x)) - V_1(x) &= \tfrac{1}{2}(x_2^+)^2 + \gamma x_1^+ - \tfrac{1}{2}x_2^2 - \gamma x_1 \\ &= \tfrac{1}{2}(-\lambda x_2)^2 - \tfrac{1}{2}x_2^2 \\ &= -\tfrac{1}{2}(1-\lambda^2)x_2^2 \\ &= -\tfrac{1}{2}(1-\lambda^2)(x_1^2 + x_2^2) = -\rho(|x|). \end{aligned}$$

Main Lyapunov theorem: example, the bouncing ball

Let us modify the Lyapunov function as, for any $x \in C \cup D \cup G(D)$,

$$V_2(x) = (1 + \theta \arctan(x_2))\, V_1(x), \quad \theta = \frac{1-\lambda^2}{\pi(1+\lambda^2)}$$

Then,

$$\frac{1}{2} V_1(x) \leq V_2(x) = (1 + \theta \arctan(x_2))\, V_1(x) \leq 2 V_1(x)$$

from which we derive that

$$\frac{1}{2}\alpha_1(|x|) \leq V_2(x) \leq 2\alpha_2(|x|).$$

Let $x \in C$ and $f \in F(x)$,

$$\langle \nabla V_2(x), f \rangle \quad = \quad 0 + \frac{\theta}{1+x_2^2}(-\gamma) V_1(x) = -\rho_1(|x|).$$

Main Lyapunov theorem: bouncing ball

Let $x \in D$, after some computations and exploiting the expression of θ

$$V_2(g(x)) - V_2(x) \quad \leq \quad -\rho_2(|x|).$$

The conditions of the Lyapunov theorem are verified by taking $\rho = \min\{\rho_1, \rho_2\}$. We conclude that $\mathcal{A} = \{(0,0)\}$ is UGpAS.

Main Lyapunov theorem: converse result

Consider

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \tag{$\mathcal{H}$}$$

If $\mathcal{A}$ is UGpAS, does it always exist a Lyapunov function V?

Theorem

If $\mathcal{A}$ is **compact** and UGpAS for system $\mathcal{H}$, which satisfies the hybrid basic conditions, then there exists a smooth Lyapunov function V, which satisfies the conditions stated previously.

Main Lyapunov theorem: converse result

Consider

$$\dot{x} \in F(x) \quad x \in C, \qquad x^+ \in G(x) \quad x \in D \qquad (\mathcal{H})$$

If $\mathcal{A}$ is UGpAS, does it always exist a Lyapunov function V?

Theorem

If $\mathcal{A}$ is **compact** and UGpAS for system $\mathcal{H}$, which satisfies the hybrid basic conditions, then there exists a smooth Lyapunov function V, which satisfies the conditions stated previously.

Main Lyapunov theorem: remarks

Often not easy to check these conditions.

No general formula, case-by-case.

$\rightarrow$ already the case for nonlinear differential/difference equations/inclusions

Main Lyapunov theorem: towards relaxed conditions

Recall

$$\begin{cases} \alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) & \forall x \in C \cup D \cup G(D) \\ \langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) & \forall x \in C,\ f \in F(x) \\ V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) & \forall x \in D,\ g \in G(x), \end{cases}$$

Relaxed conditions → easier to check (not necessarily easy ;)):

- Instead of strict decrease on flow → non-increase on flows,
- Instead of strict decrease at jumps → non-increase at jumps,
- Non-strict decrease on flows and at jumps → invariance principles

Overview

1. Stability, an intuitive treatment
2. Definition
3. Main Lyapunov theorem
4. Relaxed Lyapunov theorems and an invariance result
5. Discussions
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Relaxed Lyapunov theorems: preamble

In this section, function V is assumed to be such that

- $V : \text{dom } V \to \mathbb{R}_{\geq 0}$,
- $C \cup D \cup G(D) \subset \text{dom } V$,
- V is continuous differentiable on a open set containing C,
- There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for any $x \in C \cup D \cup G(D)$,

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}).$$

Relaxed Lyapunov theorems: non-increase at jumps

Theorem

Consider system $\mathcal{H}$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$. Suppose there exists:

- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$

such that

$$\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in C,\ f \in F(x)$$

$$V(g) - V(x) \leq 0 \qquad \forall x \in D,\ g \in G(x).$$

If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, any $(t, j) \in \operatorname{dom} \phi$, and $T \geq 0$,

$$t + j \geq T \Rightarrow t \geq \gamma_r(T) - N_r,$$

then $\mathcal{A}$ is UGpAS.

"If we flow enough, we are good."

Relaxed Lyapunov theorems: non-increase at jumps

If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_\mathcal{A} \in (0, r]$, any $(t,j) \in \operatorname{dom}\phi$, and $T \geq 0$,

$$t + j \geq T \Rightarrow t \geq \gamma_r(T) - N_r.$$

Suppose solutions have a dwell-time $\tau > 0$, i.e. there exists $\tau > 0$ units of time between two successive jump instants.

For any solution ϕ and $(t,j) \in \operatorname{dom}\phi$,

$$t \geq \tau j$$

Not exactly, because of what happens between the initial time $(0,0)$ and the first jump, so

$$\begin{aligned} t + \tau &\geq \tau j, \\ \frac{t}{\tau} + 1 &\geq j. \end{aligned}$$

Let $T \geq 0$ and $t + j \geq T$,

$$\begin{aligned} t + j &\geq T \\ t + \tfrac{t}{\tau} + 1 &\geq T \\ (1 + \tfrac{1}{\tau})t + 1 &\geq T \\ t + (1 + \tfrac{1}{\tau})^{-1} &\geq T(1 + \tfrac{1}{\tau})^{-1} \\ t &\geq \underbrace{T(1 + \frac{1}{\tau})^{-1}}_{\gamma_r(T)} - \underbrace{(1 + \frac{1}{\tau})^{-1}}_{N_r} \end{aligned}$$

Relaxed Lyapunov theorems: non-increase at jumps

If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, any $(t, j) \in \operatorname{dom}\phi$, and $T \geq 0$,

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Relaxed Lyapunov theorems: non-increase at jumps

If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, any $(t,j) \in \operatorname{dom}\phi$, and $T \geq 0$,

$$t + j \geq T \Rightarrow t \geq \gamma_r(T) - N_r.$$

Suppose solutions have a dwell-time $\tau > 0$, i.e. there exists $\tau > 0$ units of time between two successive jump instants.

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Not exactly, because of what happens between the initial time $(0,0)$ and the first jump, so

$$t + \tau \geq \tau j,$$
$$\frac{t}{\tau} + 1 \geq j.$$

Let $T \geq 0$ and $t + j \geq T$,

$$\begin{aligned}
t + j &\geq T \\
t + \tfrac{t}{\tau} + 1 &\geq T \\
(1 + \tfrac{1}{\tau})t + 1 &\geq T \\
t + (1 + \tfrac{1}{\tau})^{-1} &\geq T(1 + \tfrac{1}{\tau})^{-1} \\
t &\geq \underbrace{T(1 + \frac{1}{\tau})^{-1}}_{\gamma_r(T)} - \underbrace{(1 + \frac{1}{\tau})^{-1}}_{N_r}
\end{aligned}$$

Relaxed Lyapunov theorems: non-increase at jumps

If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, any $(t,j) \in \operatorname{dom}\phi$, and $T \geq 0$,

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Suppose solutions have a dwell-time $\tau > 0$, i.e. there exists $\tau > 0$ units of time between two successive jump instants.

For any solution ϕ and $(t,j) \in \operatorname{dom}\phi$,

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Not exactly, because of what happens between the initial time $(0,0)$ and the first jump, so

$$\begin{gathered} t + \tau \geq \tau j, \\ \frac{t}{\tau} + 1 \geq j. \end{gathered}$$

Let $T \geq 0$ and $t + j \geq T$,

$$\begin{array}{rcl} t + j & \geq & T \\ t + \frac{t}{\tau} + 1 & \geq & T \\ (1 + \frac{1}{\tau})t + 1 & \geq & T \\ t + (1 + \frac{1}{\tau})^{-1} & \geq & T(1 + \frac{1}{\tau})^{-1} \\ t & \geq & \underbrace{T(1 + \frac{1}{\tau})^{-1}}_{\gamma_r(T)} - \underbrace{(1 + \frac{1}{\tau})^{-1}}_{N_r} \end{array}$$

Relaxed Lyapunov theorems: non-increase on flow

Theorem

Consider system $\mathcal{H}$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$. Suppose there exists:

- $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite, i.e. $\rho(s) > 0$ for $s > 0$ and $\rho(0) = 0$

such that

$$\langle \nabla V(x), f \rangle \leq 0 \qquad \forall x \in C,\ f \in F(x)$$

$$V(g) - V(x) \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in D,\ g \in G(x).$$

If, for each $r > 0$, there exists $\gamma_r \in \mathcal{K}_\infty$, $N_r \geq 0$ such that for any solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$, any $(t, j) \in \operatorname{dom} \phi$, and $T \geq 0$,

$$t + j \geq T \Rightarrow j \geq \gamma_r(T) - N_r,$$

then $\mathcal{A}$ is UGpAS.

The bottom conditions is verified when solutions have an reverse (average) dwell-time.

Relaxed Lyapunov theorems: non-increase on flow, example

Bouncing ball example

We had

$$V_1(x) := \frac{1}{2}x_2^2 + \gamma x_1.$$

and

$$\begin{cases} \alpha_1(|x|) \leq V_1(x) \leq \alpha_2(|x|) \\ \langle \nabla V_1(x), f \rangle = 0 \\ V_1(g(x)) - V_1(x) \leq -\rho(|x|). \end{cases}$$

For any $r > 0$, there exists $\tau_r > 0$ such that for any solution x with $|x(0,0)| \leq r$, $\sup_t \operatorname{dom} x < \tau_r$.

Hence, for any $T \geq 0$, $t + j \geq T$ implies $j \geq T - t \geq T - \tau_r = \gamma_r(T) - N_r$.

The conditions of the relaxed theorem are verified, $\mathcal{A} = \{(0,0)\}$ is UGpAS.

Relaxed Lyapunov theorems: kind of generalization

Theorem

Consider system $\mathcal{H}$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$. Suppose there exist $\lambda_c, \lambda_d \in \mathbb{R}$ such that

$$\langle \nabla V(x), f \rangle \leq \lambda_c V(x) \qquad \forall x \in C,\ f \in F(x)$$

$$V(g) \leq e^{\lambda_d} V(x) \qquad \forall x \in D,\ g \in G(x).$$

If there exist $\gamma, M > 0$ such that for any solution x, and any $(t, j) \in \operatorname{dom} x$,

$$\lambda_c t + \lambda_d j \leq M - \gamma(t + j),$$

then $\mathcal{A}$ is UGpAS.

Idea of the proof: for any solution x and $(t, j) \in \operatorname{dom} x$, by integration (comparison principle)

$$V(x(t, j)) \leq e^{\lambda_c t + \lambda_d j} V(x(0, 0))$$

using $\lambda_c t + \lambda_d j \leq M - \gamma(t + j)$, we derive

$$V(x(t, j)) \leq e^{M - \gamma(t+j)} V(x(0, 0)),$$

from which we can derive $\mathcal{KL}$-stability of $\mathcal{A}$.

Relaxed Lyapunov theorems: kind of generalization

Recall

$$\langle \nabla V(x), f \rangle \leq \lambda_c V(x) \qquad \forall x \in C,\ f \in F(x)$$

$$V(g) \leq e^{\lambda_d} V(x) \qquad \forall x \in D,\ g \in G(x).$$

We can always modify a Lyapunov function V such that its increasing/decreasing properties are exponential as above.

Relaxed Lyapunov theorems: invariance principle

Still, to find a positive definite function ρ such that

$$\langle \nabla V(x), f \rangle \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in C,\ f \in F(x)$$

or

$$V(g) \leq -\rho(|x|_{\mathcal{A}}) \qquad \forall x \in D,\ g \in G(x)$$

is not always easy.

We may then rely on so-called invariance principle, we mean here LaSalle-Barbasin-Krasovkii type of results.

General statements in [Goebel et al., 2012].

We are going to see a particular useful invariance principle published in:

- A. Seuret, C. Prieur, S. Tarbouriech, A.R. Teel, L. Zaccarian, *A nonsmooth hybrid invariance principle applied to robust event-triggered design*, IEEE Transactions on Automatic Control, 2018.

Relaxed Lyapunov theorems: invariance principle

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Relaxed Lyapunov theorems: invariance principle

Still, to find a positive definite function ρ such that

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Relaxed Lyapunov theorems: invariance principle

Let $\mathcal{A} \subset \mathbb{R}^n$ be a **compact** set satisfying $G(\mathcal{A} \cap D) \subset \mathcal{A}$.

Theorem

Consider system $\mathcal{H}$ and suppose the following holds

$$\langle \nabla V(x), f \rangle \leq 0 \qquad \forall x \in C \backslash \mathcal{A},$$

$$V(g) - V(x) \leq 0 \qquad \forall x \in D \backslash \mathcal{A}, \ g \in G(x).$$

and no complete solution keeps V constant and nonzero, i.e. no complete solution x exists and satisfies $V(x(t,j)) = V(x(0,0)) \neq 0$ for all $(t,j) \in \operatorname{dom} x$.

Then $\mathcal{A}$ is UGAS.

Overview

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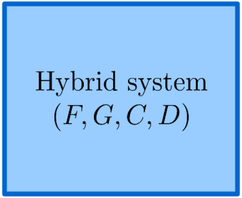
Discussions: indirect Lyapunov theorems

For differential/difference equations, we also have Lyapunov indirect theorems
$\rightarrow$ linearize the system around a point / analyse the stability of the linearized model / conclude local stability properties for the original system

Such results are provided in Chapter 9 of [Goebel et al., 2012]

Discussions: other stability properties

In this course, we concentrate on internal stability



Input-output properties

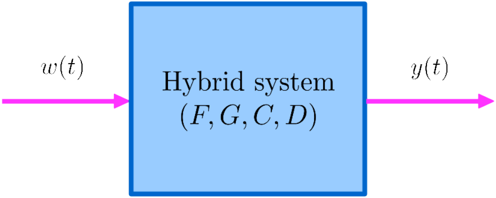
- $\mathcal{L}_p$-stability,
- input-to-state stability, input-to-output stability etc.
- dissipativity.

Other stability related results:

- Incremental stability, contraction etc.
- Small-gain theorems

Discussions: other stability properties

In this course, we concentrate on internal stability



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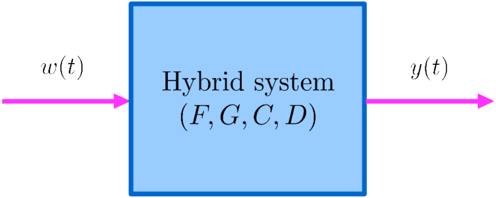

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Overview

1. Stability, an intuitive treatment
2. Definition
3. Main Lyapunov theorem
4. Relaxed Lyapunov theorems and an invariance result
5. Discussions
6. Summary

Summary

- Set stability
- Definition of stability
- Nominal robustness
- Lyapunov theorems
- Relaxed version and an invariance result

Summary: references

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