# Self-Triggered Output Feedback Control for Perturbed Linear Systems

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**Abstract:** In this work we propose a Self-Triggered Control (STC) strategy for linear time-invariant (LTI) systems subject to bounded disturbances, using LTI discrete-time dynamic output-feedback. The STC logic computes worst-case triggering times from available information, based on a Periodic Event Triggered Control (PETC) triggering function. In the case of no perturbation and full state information, the discrete times can be determined exactly, defining unions of cones in the state space. When bounded disturbances are present, we compute worst-case triggering times and their associated state-space regions. If full state information is also not available, we use a special observer to compute the worst-case triggering times, yielding an STC logic that only needs the current system output and the controller state. For all cases, we provide sufficient conditions for stability and $\mathcal{L}_2$-gain from disturbance to output.

*Keywords:* Control systems, digital control, linear systems, bounded disturbances, self-triggered control, networked control systems.

## 1. INTRODUCTION

In Networked Control Systems (NCSs), communication constraints can impose limits to the number of devices sharing a network. Promising to reduce the amount of transfered control data, aperiodic sampling methods like Event-Triggered Control (ETC) and Self-Triggered Control (STC) have been gaining increased attention, especially since the results obtained over the past decade in, e.g., Tabuada (2007); Anta and Tabuada (2008); Mazo Jr. et al. (2010); Fiter et al. (2012); Heemels et al. (2013).

Through a simple triggering mechanism, ETC only updates sensors data and actuator commands when certain conditions are met. While effective in reducing the number of communications, ETC has little communication predictability. Such predictability is critical for NCSs because communications need to be properly scheduled so as to avoid conflicts and lose control performance. STC was proposed aiming at this predictability. In STC, the controller decides the next sampling instant based on the estimate of when an underlying triggering condition would occur. Unfortunately, even for linear systems, estimating the triggering time involves solving a transcendental equation (Mazo Jr. et al., 2010), so existing solutions are generally conservative when compared to the ETC equivalent, i.e., sampling occurs more often. Another drawback is that the implementation loses its original simplicity and more computational power is needed for the time computations.

Another approach to deal with communication predictability in the ETC/STC context is to design a scheduler for ETC. A promising approach is by abstracting the closed-loop system by Timed Automata, whose discrete states are partitions of the original state-space. Kolarijani and Mazo Jr (2016) have been successful in defining these partitions for linear time-invariant (LTI) systems. However, the number of resulting polytopes grows exponentially with the plant's state space dimension.

When systems are subject to disturbance, some important properties of ETC are lost if the well established relative triggering is used. In particular, the event separation property can be lost when the state norm is small enough, i.e., the Zeno effect can occur (Borgers and Heemels, 2014). When output-feedback is used, Zeno can occur even in the absence of disturbances. One possible solution to avoid Zeno is using Periodic ETC (PETC), proposed by Heemels et al. (2013), where the controller runs in a periodic fashion – in that way, event separation of at least one sample time can be ensured. In his thesis, Fu (2018) derives an abstraction for PETC for linear systems subject to bounded disturbances, in the same philosophy as Kolarijani et al. (2015); Kolarijani and Mazo Jr (2016). Such abstraction suffers from the same curse of dimensionality and introduces significant conservatism when disturbances are taken into account. Also, for the output-feedback case, the abstraction still relies on full-state information.

We set ourselves the task of finding improved timing prediction for ETC of LTI systems subject to bounded disturbances. We start from the STC implementation of the PETC for the unperturbed state-feedback case in Donkers (2011). Our contribution is expanding it for the perturbed case and for the output-feedback case. In both cases, the corresponding worst-case triggering times can be used either as an STC or as a worst-case estimate for the next triggering time of a PETC. For the perturbed case, our solution results in a state-space partition for lower bounds of triggering times that is less conservative than the ones existing in the literature, such as in Fu (2018). Our partitioning also does not suffer from the curse of

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dimensionality as the number of sets is a function of the time steps, growing linearly with the selected precision.

STC for perturbed LTI systems has also been considered in Mazo Jr. et al. (2010). While the authors succeed in proposing an STC that renders the closed loop *exponentially input to state stable* (EISS), the resulting EISS-gain from disturbance to state is often very high. In a sense, our work on perturbed state-feedback STC is also similar to the "State-Dependent Sampling" (SDS) proposed in Fiter et al. (2012), but instead of using the same regions as in the unperturbed case, we propose new ones. Compared to it, our approach is significantly simpler, as their solution requires solving a significant number of LMIs and line searches (at least 400 in the provided 2-state example). For output feedback, the literature is more scarce. Almeida et al. (2014) have proposed STC for output-feedback, but they constrained their solution to observer-based state feedback. In this work we consider a more general form of linear dynamic output-feedback controller.

### 1.1 Notation

We denote as $\mathbb{N}_0$ the set of natural numbers including 0 and $\mathbb{N}$ the same set without it. For a vector $x \in \mathbb{R}^n$ we denote by $\|x\| := \sqrt{x^T x}$ its 2-norm. For a matrix $A \in \mathbb{R}^{n\times m}$ we denote by $A^T$ its transpose, by $\lambda_{\max}(A)$ its maximum eigenvalue, by $|A|$ its 2-induced norm and by $\mathrm{Tr}\,(A)$ its trace. For a symmetric matrix $A$ described in blocks, we may use $\star$ to denote blocks that can be induced by symmetry. Also, for any matrix $A$, a row index set $\mathcal{I} \subseteq \{1, ..., n\}$ and a column index set $\mathcal{J} \subseteq \{1, ..., m\}$ we denote $A|_{\mathcal{I},\mathcal{J}}$ the sub-matrix of $A$ indexed by those sets. If $\mathcal{I} = \{1, \ldots, n\}$ or $\mathcal{J} = \{1, \ldots, m\}$ we use $A|_{\mathcal{I},\bullet}$ or $A|_{\bullet,\mathcal{J}}$, respectively. For a symmetric square matrix $S \in \mathbb{R}^{n\times n}$, we use $S \succ 0$ if it is positive-definite and $S \succeq \mathbf{0}$ if it is semi-positive definite. For a signal $\xi : \mathbb{R}^+ \to \mathbb{R}^n$, $\|\xi(t)\|_{\mathcal{L}_2} := \sqrt{\int_0^\infty \xi(\tau)^2 d\tau}$ denotes its $\mathcal{L}_2$-norm and $\|\xi(t)\|_{\mathcal{L}_\infty} := \operatorname{ess\,sup}_t |\xi(t)|$ denotes its $\mathcal{L}_\infty$-norm.

## 2. PROBLEM DEFINITION

Consider an LTI system of the form:

$$\begin{cases} \dot{\xi}_p(t) = A_p\xi_p(t) + B_p\hat{\upsilon}(t) + E\omega(t) \\ \psi(t) = C_p\xi_p(t), \end{cases} \tag{1}$$

where $\xi_p(t) \in \mathbb{R}^{n_p}$ is the plant state, $\hat{\upsilon}(t) \in \mathbb{R}^{n_\upsilon}$ is its input, $\omega(t) \in \mathbb{R}^{n_\omega}$ is the unknown disturbance and $\psi(t) \in \mathbb{R}^{n_\psi}$ is the plant's output. $A_p$, $B_p$, $E$ and $C_p$ are constant matrices of appropriate dimensions. The plant is controlled with discrete-time output feedback as in Heemels et al. (2013); Fu (2018):

$$\begin{cases} \xi_c(k+1) = A_c\xi_c(k) + B_c\hat{\psi}(k) \\ \upsilon(k) = C_c\xi_c(k) + D_c\hat{\psi}(k), \end{cases} \tag{2}$$

where $\xi_c(k) \in \mathbb{R}^{n_c}$ is the controller state, $\upsilon(k) \in \mathbb{R}^{n_\upsilon}$ is the controller command and $\hat{\psi}(k) \in \mathbb{R}^{n_\psi}$ is its available input. $A_c$, $B_c$, $C_c$ and $D_c$ are constant matrices of appropriate dimensions.

The controller runs every $h$ time units, so we define $t_k = kh$ as the time instant at which the controller runs its $k$-th update. A sample-and-hold mechanism is used between controller and plant, such that, for a sampling sequence $\mathcal{K} := \{k_b | b \in \mathbb{N}_0\}$ we have that $\hat{\upsilon}(t) = \upsilon(k_b)$ for $t \in [hk_b, hk_{b+1})$ and $\hat{\psi}(k) = \psi(hk_b)$ for $k \in [k_b, k_{b+1})$.

On a self-triggered control strategy, every time the output vector $\psi$ is updated and $\upsilon$ is sent to the actuator, the controller decides at which instant the next update will be required. On the discrete-time controller considered, if the controller updates its data at instant $k$, then it will decide a value $\Delta k \in \mathbb{N}$ such that data will be updated again at $k + \Delta k$. This is when the controller requests data from the sensors, updates its output and sends it to the actuators. We can then define the sequence of triggering instants on a self-triggered control strategy as:

$$\mathcal{K}_s := \{k_b | b \in \mathbb{N}_0, k_0 = 0, k_{b+1} = k_b + \Delta k_b\}\,. \tag{3}$$

Throughout this paper, the following assumptions hold:

*Assumption 1.* Disturbances satisfy $\omega \in \mathcal{L}_2$ and the pair $(A_p, C_p)$ in (1) is observable.

We now define our main problem:

*Problem 2.* Given a plant model (1) and a controller model (2) with time step $h$, design an algorithm that computes $\Delta k_b$ at instant $t_{k_b}$, such that the closed loop system sampled under the sequence (3) is input-to-state stable.

## 3. SELF-TRIGGERED CONTROL FOR PERTURBED SYSTEMS

In this section we revisit a PETC strategy and design a self-triggering rule that guarantees stability for systems with bounded disturbances $\omega(t)$.

### 3.1 Periodic Event-Triggered Control

Let $\zeta^T := \begin{bmatrix}\psi^T & \upsilon^T\end{bmatrix}$ and $\hat{\zeta}^T := \begin{bmatrix}\hat{\psi}^T & \hat{\upsilon}^T\end{bmatrix}$. The centralized version of the output-feedback PETC proposed in Heemels et al. (2013) is as follows:

$$\hat{\zeta}(t_k) = \begin{cases} \zeta(t_k), & \text{if } \|\zeta(t_k) - \hat{\zeta}(t_{k-1})\| > \sigma\|\zeta(t_k)\| \\ \hat{\zeta}(t_{k-1}), & \text{otherwise.} \end{cases} \tag{4}$$

The event condition for (4) and several other PETC mechanisms can be reformulated as a quadratic form (Heemels et al., 2013), defining the following event sequence:

$$\mathcal{K}_b := \left\{k_b | b \in \mathbb{N}_0, \chi^T(t_{k_b})Q\chi(t_{k_b}) > 0\right\}, \tag{5}$$

where $\chi^T(t_k) := \begin{bmatrix}\xi^T(t_k) & \hat{\zeta}^T(t_k)\end{bmatrix}, \xi^T(t_k) := \begin{bmatrix}\xi_p^T(t_k) & \xi_c^T(k)\end{bmatrix}$, with (Fu, 2018):

$$Q = \begin{bmatrix}\bar{Q}_1 & \bar{Q}_2 \\ \bar{Q}_2^T & \bar{Q}_3\end{bmatrix}, \tag{6}$$

$$\bar{Q}_1 = \begin{bmatrix}\varsigma C_p^T C_p & \mathbf{0} \\ \mathbf{0} & \varsigma C_c^T C_c\end{bmatrix},\ \bar{Q}_2 = \begin{bmatrix}-C_p^T & \mathbf{0} \\ \varsigma C_c^T D_c & -C_c^T\end{bmatrix}, \\ \bar{Q}_3 = \begin{bmatrix}I + \varsigma D_c^T D_c & -D_c^T \\ -D_c & I\end{bmatrix}. \tag{7}$$

Matrices $I$ and $\mathbf{0}$ are respectively the identity and the zero matrix, both with appropriate dimensions, and $\varsigma := 1 - \sigma^2$. The vector $\hat{\zeta}(t_k)$ can be expressed as a function of the current state after the last trigger:

$$\hat{\zeta}(t_k) = C_E\xi(t_k),\ C_E = \begin{bmatrix}C_p & \mathbf{0} \\ D_cC_p & C_c\end{bmatrix}$$

Since the system is LTI, the state after $\Delta k$ samples can be written as $\xi(t_{k+\Delta k}) = M(\Delta k)\xi(t_k) + \theta(\Delta k)$, with:

$$M(\Delta k) = \begin{bmatrix} M_1(\Delta k) \\ M_2(\Delta k) \end{bmatrix}, \ \theta(\Delta k) = \begin{bmatrix} \theta_1(\Delta k) \\ \mathbf{0} \end{bmatrix}.$$

$$\begin{aligned} M_1(\Delta k) &= [I \ \mathbf{0}] + \int_0^{h\Delta k} e^{A_p s} ds \left(A_p [I \ \mathbf{0}] + B_p [D_c C_p \ C_c]\right), \\ M_2(\Delta k) &= A_c^{\Delta k} [\mathbf{0} \ I] + \sum_{i=0}^{\Delta k-1} A_c^i B_c [C_p \ \mathbf{0}], \\ \theta_1(\Delta k) &= \int_0^{\Delta k h} e^{A_p(\Delta k h - s)} E\omega(s) ds. \end{aligned} \tag{8}$$

Therefore the triggering condition can be written as:

$$\begin{aligned} \underline{\Delta k}(\xi(t_k)) &= \inf \left\{ \Delta k \in \mathbb{N} \,\middle|\, \chi(t_{k+\Delta k})^T Q \chi(t_{k+\Delta k}) > 0 \right\}, \\ \chi(t_{k+\Delta k}) &= \begin{bmatrix} M(\Delta k)\xi(t_k) + \theta(\Delta k) \\ C_E \xi(t_k) \end{bmatrix}. \end{aligned} \tag{9}$$

If we design an STC logic that triggers no later than the PETC, i.e., computing $\Delta k \leq \underline{\Delta k}(\xi(t_k))$, closed loop stability and $\mathcal{L}_2$-gain can be verified with the following corollary to Theorem V.2 in Heemels et al. (2013):

*Corollary 3.* Consider the following non-deterministic impulsive model:

$$\begin{aligned} \begin{bmatrix} \dot{\chi}(t) \\ \dot{\tau}(t) \end{bmatrix} &= \begin{bmatrix} \bar{A}\chi(t) + \bar{B}\omega(t) \\ 1 \end{bmatrix}, \text{ when } \tau(t) \in [0, h] \\ \begin{bmatrix} \chi^+(t) \\ \tau^+(t) \end{bmatrix} &= \begin{cases} \begin{bmatrix} J_1 \chi(t) \\ 0 \end{bmatrix}, \text{ when } t = h, \\ \begin{bmatrix} J_2 \chi(t) \\ 0 \end{bmatrix}, \text{ when } t = h, \chi^T(t) Q \chi(t) \leq 0, \end{cases} \\ \psi(t) &= \bar{C}\chi(t), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_p & \mathbf{0} & \mathbf{0} & B_p \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \bar{B} = \begin{bmatrix} E \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \ \bar{C} = [C_p \ \mathbf{0} \ \mathbf{0} \ \mathbf{0}], \\ J_1 &= \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_c C_p & A_c & \mathbf{0} & \mathbf{0} \\ C_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_c & D_c & \mathbf{0} \end{bmatrix}, \ J_2 = \begin{bmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_c & B_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I \end{bmatrix}, \end{aligned} \tag{11}$$

with $\bar{A}$, $\bar{B}$, $J_1$ and $J_2$ being partitioned according to $\chi^T = \begin{bmatrix} \xi_p^T & \xi_c^T & \hat{\psi}^T & \hat{\nu}^T \end{bmatrix}$. Let $\rho > 0, \gamma > 0$ and consider also

$$\begin{aligned} H &:= \begin{bmatrix} \bar{A} + \rho I & \bar{B}\bar{B}^T \\ -\gamma^2 \bar{C}^T \bar{C} & -(\bar{A} + \rho I)^T \end{bmatrix}, \\ G(t) &:= e^{-Ht} = \begin{bmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{bmatrix}. \end{aligned} \tag{12}$$

Assume $G_{11}(t)$ in invertible for all $t \in [0, h]$ [1]. Denote $\bar{G}_{11} := G_{11}(h)$, $\bar{G}_{12} := G_{12}(h)$ and $\bar{G}_{21} := G_{21}(h)$. Suppose there exists a symmetric matrix $P_h \succ 0$ and a scalar $\mu \geq 0$ such that

$$\begin{aligned} &\begin{bmatrix} P_h & J_1^T \bar{G}_{11}^{-T} P_h \bar{S} & J_1^T \Phi(P_h) \\ \star & I - \bar{S}^T P_h \bar{S} & \mathbf{0} \\ \star & \star & \Phi(P_h) \end{bmatrix} \succ 0, \\ &\begin{bmatrix} P_h + \mu Q & J_2^T \bar{G}_{11}^{-T} P_h \bar{S} & J_2^T \Phi(P_h) \\ \star & I - \bar{S}^T P_h \bar{S} & \mathbf{0} \\ \star & \star & \Phi(P_h) \end{bmatrix} \succ 0, \end{aligned} \tag{13}$$

with $\Phi(P_h) := \bar{G}_{11}^{-1} P_h \bar{G}_{11}^{-1} + \bar{G}_{21}\bar{G}_{11}^{-1}$ and $\bar{S}$ satisfying $\bar{S}\bar{S}^T = -\bar{G}_{11}^{-T}\bar{G}_{12}$. Then the PETC implementation in (4) is Globally Exponential Stable (GES) with decay rate $\rho$ when $\omega(t) \equiv 0$ and has an $\mathcal{L}_2$-gain from $\omega$ to $\psi$ smaller or equal than $\gamma$. Moreover, *any logic that triggers no later than such a PETC logic has the same stability and $\mathcal{L}_2$-gain properties.*

**Proof.** (Sketch) We kindly refer the reader to the proof of the original theorem in (Heemels et al., 2013, Theorem V.2), where a similar set of Linear Matrix Inequalities (LMIs) are derived as sufficient conditions for the stability and $\mathcal{L}_2$-gain properties of the PETC. From that, consider a centralized version of the output feedback, constraining the $J$ matrices to $J_1$ and $J_2$ in (11). The main change with respect to the original Theorem is to consider that triggering may happen earlier, so the triggering function $\chi^T(t)Q\chi(t)$ may still be smaller than zero. Therefore, we allow the jumps in (10) according to $J_1$ at any situation, not only when the triggering function is positive. Then, while the second LMI in (13) uses the S-procedure to encode the triggering function being non-positive, the first LMI assumes nothing related to the triggering function, and the term $P_h$ comes alone in the first block. [2]

### *3.2 Computation of self triggering times*

Based on Eq. (9) we start by constructing a self-triggered logic depending solely on the state value $\xi(k_b)$ at each triggering time. To simplify the notation, we hereafter denote $x := \xi(t_{k_b})$ as the current state available to the controller and use subscripts for the sample dependence on the matrices: $M_{\Delta k} := M(\Delta k)$ and $\theta_{\Delta k} := \theta_1(\Delta k)$.

We start by considering zero disturbance. In this case, the PETC has an equivalent STC implementation as noted by Donkers (2011). The minimum triggering time in (9) can be written as:

$$\underline{\Delta k}(x) = \inf \left\{ \Delta k \in \mathbb{N} \,\middle|\, x^T Q_{\Delta k} x > 0 \right\}, \tag{14}$$

where

$$Q_{\Delta k} = \begin{bmatrix} M_{\Delta k} \\ C_E \end{bmatrix}^T Q \begin{bmatrix} M_{\Delta k} \\ C_E \end{bmatrix}. \tag{15}$$

We denote by $\mathcal{R}_{\Delta k} \subseteq \mathbb{R}^{n_p+n_c}$ the set of states $x$ such that triggering occurs according to (14). These subspaces can be calculated as:

$$\begin{aligned} \mathcal{R}_1 &= \{x \in \mathbb{R}^{n_p+n_c} | x^T Q_1 x > 0\} \\ \mathcal{R}_{\Delta k} &= \{x \in \mathbb{R}^{n_p+n_c} | x^T Q_{\Delta k} x > 0\} \setminus \bigcup_{l=1}^{\Delta k-1} \mathcal{R}_l \end{aligned} \tag{16}$$

With a designed maximum triggering time $\overline{\Delta k}$, the values of $Q_{\Delta k}, \Delta k = 1, ..., \overline{\Delta k}$, can be computed off-line. Hence, checking at which region $\mathcal{R}_{\Delta k}$ the state belongs takes at most $\overline{\Delta k}$ operations.

Now we consider the effect of disturbances. We start from the following assumption (Fu, 2018):

*Assumption 4.* The disturbance $\omega(t)$ satisfies $\omega(t) \in \mathcal{L}_\infty$. Besides, an upper bound $W > 0, W \in \mathbb{R}$ for the latter is known, i.e., $W \geq \|\omega(t)\|_{\mathcal{L}_\infty}$.

[1] It always holds for sufficiently small $h$ (Heemels et al., 2013)

[2] The first LMI is a sufficient condition for the periodic controller to have such stability and $\mathcal{L}_2$-gain properties.

This allows us to determine a bound to $\|\theta_{\Delta k}\|$ (Fu, 2018):

$$\|\theta_{\Delta k}\| \leq \int_0^{h\Delta k} e^{\lambda_{\max}\left(\frac{A_p^T+A_p}{2}\right)s} ds|E|W = \Theta_{\Delta k}. \quad (17)$$

Now we can design a triggering condition that triggers no later than the one in (4). Recall that it can be expressed in the form (9), which in turn can be expanded as:

$$x^T Q_{\Delta k} x + 2x^T \begin{bmatrix} M_{\Delta k} \\ C_E \end{bmatrix}^T \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2^T \end{bmatrix} \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \theta_{\Delta k} + \theta_{\Delta k}^T \bar{Q}_{1p} \theta_{\Delta k} > 0, \quad (18)$$

where $\bar{Q}_{1p} := \bar{Q}_1|_{\mathcal{N},\mathcal{N}}, \mathcal{N} := \{1, ..., n_p\}$. Using the bound in (17) we define a new triggering logic of the form:

$$x^T Q_{\Delta k} x + 2\|F_{\Delta k}^T x\|\Theta_{\Delta k} + c_{\Delta k} > 0, \quad (19)$$

where $F_{\Delta k} := [I\ \mathbf{0}](\bar{Q}_1 M_{\Delta k} + \bar{Q}_2 C_E)$ and $c_{\Delta k} := |\bar{Q}_{1p}|\Theta_{\Delta k}^2$. The left-hand side (LHS) of (19) is by construction always greater than or equal to the LHS of (18). Therefore, (19) constitutes a conservative triggering logic with respect to the disturbances which only depends on the state. It can be implemented as an STC with the following algorithm:

*Algorithm 1.* Computation of triggering times.

$\Delta k \leftarrow 1$
$s \leftarrow x^T Q_{\Delta k} x + 2\|F_{\Delta k}^T x\|\Theta_{\Delta k} + c_{\Delta k}$
**while** $(s \leq 0) \wedge (\Delta k \leq \overline{\Delta k})$ **do**
  $\Delta k \leftarrow \Delta k + 1$
  $s \leftarrow x^T Q_{\Delta k} x + 2\|F_{\Delta k}^T x\|\Theta_{\Delta k} + c_{\Delta k}$
**end while**

This algorithm can be used for the unperturbed case by taking $\Theta_{\Delta k} = 0, \forall \Delta k \in \{1, ..., \overline{\Delta k}\}$. Notice that $F_{\Delta k}$ and $c_{\Delta k}$ can be computed off-line. For the state-space partitioning, we can compute the sets $\mathcal{S}_{\Delta k}$ as $\mathcal{S}_1 = \{x \in \mathbb{R}^{n_p+n_c} | x^T Q_1 x + 2\|F_1^T x\|\Theta_1 + c_1 > 0\}$ and $\mathcal{S}_{\Delta k} = \{x \in \mathbb{R}^{n_p+n_c} | x^T Q_{\Delta k} x + 2\|F_{\Delta k}^T x\|\Theta_{\Delta k} + c_{\Delta k} > 0\} \setminus \bigcup_{l=1}^{\Delta k-1} \mathcal{S}_l$.

*Remark 5.* Since $c_{\Delta k} > 0$, the LHS of (19) is positive for $x = 0$ (and for sufficiently small $x$). Therefore, this logic tends to yield periodic control, with $\Delta k = 1$ as the state approaches the origin.

## 4. OUTPUT-BASED SELF-TRIGGERED CONTROL

Up to this point, we have defined self-triggered policies based on the last available state. While they work for the output feedback case, their implementation still relies on some information of the full state; namely, in which of the regions $\mathcal{R}_{\Delta k}$ or $\mathcal{S}_{\Delta k}$ they belong. This hinders its applicability to output-feedback. In this Section, we design an STC strategy that relies solely on the plant output $\hat{\psi}(k_b)$ and the controller states $\xi_c(k_b)$. However, we start by noting that some information on the unmeasured subspace of the state space is necessary to sample less frequently than periodically with $\Delta k = 1$. We can see that by segregating the contribution of the unmeasured component of the states to the function $\chi^T(t) Q \chi(t)$ ignoring disturbances. Since the plant is assumed observable, take the plant in its normal observable form. Hence, $C_p = [I\ \mathbf{0}]$ and we can set $\xi(t)^T := \left[\psi(t)^T\ \bar{\psi}(t)^T\right]$, where $\bar{\psi}(t) \in \mathbb{R}^{n_p - n_\psi}$ is the vector of unmeasured states at time $t$. Assume we have an estimate $\bar{y}_0$ for $\bar{y} := \bar{\psi}(t_{k_b})$ and denote the estimate residue $r_{\bar{y}} := \bar{y} - \bar{y}_0$. Note that $x = \tilde{x} + [\mathbf{0}\ I\ \mathbf{0}]^T r_{\bar{y}}$. From (4), we can rewrite the condition in (14) as:

$$\tilde{x}^T Q_{\Delta k} \tilde{x} + 2\tilde{x}^T Q_{\Delta k}|_{\bullet,\mathcal{J}} r_{\bar{y}} + r_{\bar{y}}^T Q_{\Delta k}|_{\mathcal{J},\mathcal{J}} r_{\bar{y}} > 0, \quad (20)$$

where $\tilde{x}^T = \left[y^T\ \bar{y}_0^T\ x_c^T\right]$, $x_c := \xi_c(k_b)$ and $\mathcal{J} = \{n_\psi + 1, \ldots, n_p\}$. From (7) and (15), we have that $Q_{\Delta k}|_{\mathcal{J},\mathcal{J}} = (1-\sigma^2) M_{\Delta k}|_{\mathcal{I},\mathcal{J}}^T M_{\Delta k}|_{\mathcal{I},\mathcal{J}} \succeq \mathbf{0}$, with $\mathcal{I} = \{1, \ldots, n_\psi\}$. Therefore, unless $M_1|_{\mathcal{I},\mathcal{J}} = \mathbf{0}$, there will always be $r_{\bar{y}}$, large enough, such that the LHS in (20) is positive.

The observation above hints that bounds are needed for the unmeasured subspace, even without disturbances. Hence, we will later build an observer that computes bounds to $r_{\bar{y}}$. For now, we assume an ellipsoidal bound is known to propose the following Lemma:

*Lemma 6.* Consider the triggering function $\chi(t_{k_b})^T Q \chi(t_{k_b})$ in (5), with the triggering logic defined in (4). Let Assumption 4 hold. Assume there is a matrix $H_b := H(k_b)$ and an estimate $\bar{y}_0$ such that $r_{\bar{y}}^T H_b^{-1} r_{\bar{y}} \leq 1$. Then the function

$$\eta(\tilde{x}, H_b, \Delta k) := \tilde{x}^T Q_{\Delta k} \tilde{x} + 2\|F_{\Delta k}^T \tilde{x}\|\Theta_{\Delta k} + c_{\Delta k} + 2\sqrt{\tilde{x}^T Q_{\Delta k}|_{\bullet,\mathcal{J}} H_b Q_{\Delta k}|_{\bullet,\mathcal{J}}^T \tilde{x}} + \lambda_{\max}\left(H_b Q_{\Delta k}|_{\mathcal{J},\mathcal{J}}\right) + 2\Theta_{\Delta k}\sqrt{\lambda_{\max}\left(Q_{\Delta k}|_{\mathcal{J},\mathcal{N}}^T H_b Q_{\Delta k}|_{\mathcal{J},\mathcal{N}}\right)} \quad (21)$$

is an upper bound to $\chi(t_{k_b})^T Q \chi(t_{k_b})$.

**Proof.** We apply a similar procedure to the one described in Sec. 3.2 and bound the individual terms of the triggering function expanded in terms of $\tilde{x}, r_{\bar{y}}$ and $\theta_{\Delta k}$:

$$\tilde{x}^T Q_{\Delta k} \tilde{x} + 2\tilde{x}^T F_{\Delta k}^T \theta_{\Delta k} + \theta_{\Delta k}^T \bar{Q}_{1p} \theta_{\Delta k} + 2\tilde{x}^T Q_{\Delta k}|_{\bullet,\mathcal{J}} r_{\bar{y}} + r_{\bar{y}}^T Q_{\Delta k}|_{\mathcal{J},\mathcal{J}} r_{\bar{y}} + 2r_{\bar{y}}^T Q_{\Delta k}|_{\mathcal{J},\mathcal{N}} \theta_{\Delta k}.$$

The first three terms are bounded like in Sec. 3.2. The remaining bounds can be derived by using the transformation $r_{\bar{y}} = Sr$, with $S$ satisfying $SS^T = H_b$. Then $r^T r \leq 1$. We therefore use matrix norms and eigenvalues to bound each term. The remaining terms are $\tilde{x}^T Q_{\Delta k}|_{\bullet,\mathcal{J}} Sr \leq \|\tilde{x}^T Q_{\Delta k}|_{\bullet,\mathcal{J}} S\|$, $r^T S^T Q_{\Delta k}|_{\mathcal{J},\mathcal{J}} Sr \leq \lambda_{\max}(S^T Q_{\Delta k}|_{\mathcal{J},\mathcal{J}} S)$; and $r_{\bar{y}}^T Q_{\Delta k}|_{\mathcal{J},\mathcal{N}} \theta_{\Delta k} \leq \Theta_{\Delta k}\left|S^T Q_{\Delta k}|_{\mathcal{J},\mathcal{N}}\right|$. Computing each term replacing $SS^T$ with $H_b$ and using the property of invariance to product order of the eigenvalue function provides the expression (21), which requires no explicit factorization of the matrix $H_b$.

Finally we can use this bound to establish a self-triggering logic for the case with incomplete state information and disturbances:

*Theorem 7.* Let the assumptions and conditions in Lemma 6 and Corollary 3 hold for given $\rho$ and $\gamma$. Then the triggering logic

$$\underline{\Delta k} = \inf\{\Delta k \in \{1, ..., \overline{\Delta k}\} : \eta(\tilde{x}, H_b, \Delta k) > 0\} \quad (22)$$

renders the closed loop system (1), (2) GES with decay rate $\rho$ and $\mathcal{L}_2$-gain smaller or equal than $\gamma$.

**Proof.** Since the $h$ superiorly bounds the PETC triggering function, the STC logic will trigger no later than when the PETC would. Therefore the stability and $\mathcal{L}_2$-gain properties are inherited from Corollary 3.

Now we design the observer that generates the ellipsoidal bounds defined by the matrix $H_b$. Based on (8), for a given triggering time $\Delta k$ we have:

$$\bar{\psi}(k_{b+1}) = M_{\Delta k}|_{\mathcal{J},\bar{\mathcal{J}}} \begin{bmatrix} \hat{\psi}(k_b) \\ \xi_c(k_b) \end{bmatrix} + M_{\Delta k}|_{\mathcal{J},\mathcal{J}} \bar{\psi}(k_b) + [\mathbf{0}\ I]\,\theta_{\Delta k},$$

where $\bar{\mathcal{J}} \coloneqq \{1, \ldots, n_p+n_c\} \backslash \mathcal{J}$. Since the close loop system using Theorem 7 is stable provided that the bounds are valid, we can use an open loop observer of the form

$$\bar{\psi}_0(k_{b+1}) = M_{\Delta k}|_{\mathcal{J},\bar{\mathcal{J}}} \begin{bmatrix} \hat{\psi}(k_b) \\ \xi_c(k_b) \end{bmatrix} + M_{\Delta k}|_{\mathcal{J},\mathcal{J}} \bar{\psi}_0(k_b). \quad (23)$$

The remaining task is how to initialize and evolve the ellipsoids $H_b$. For the initialization, some information on the initial norm of the unmeasured states is needed. We hence start with the following assumption:

*Assumption 8.* A value $R \in \mathbb{R}, R \geq 0$, is known such that $||\bar{\psi}(0)|| \leq R$.

The condition above can be rewritten as

$$(\bar{\psi}(0) - \bar{\psi}_0(0))^T H_0^{-1} (\bar{\psi}(0) - \bar{\psi}_0(0)) \leq 1,$$

with $H_0 = R^2 I$ and $\bar{\psi}_0(0) = \mathbf{0}$.

For the evolution of $H_b$ we provide the following Theorem.

*Theorem 9.* Let $\delta_{b+1} \coloneqq \bar{\psi}(k_{b+1}) - \bar{\psi}_0(k_{b+1})$ and $N_{\Delta k} \coloneqq M_{\Delta k}|_{\mathcal{J},\mathcal{J}}$. Assume $r_{\bar{y}}^T H_b^{-1} r_{\bar{y}} \leq 1$ and $\| [\mathbf{0}\ I]\, \theta_{\Delta k} \| \leq \Theta_{\Delta k}$. Denote by $H_\theta \coloneqq \Theta_{\Delta k}^2 I$. The update law

$$H_{b+1} = \left(1 + p_b^{-1}\right) N_{\Delta k}^T H_b N_{\Delta k} + (1 + p_b) H_\theta, \quad (24a)$$

$$p_b = \sqrt{\mathrm{Tr}\left(N_{\Delta k}^T H_b N_{\Delta k}\right)} \left(\sqrt{\mathrm{Tr}\left(H_\theta\right)}\right)^{-1}, \quad (24b)$$

satisfies $\delta_{b+1}^T H_{b+1}^{-1} \delta_{b+1} \leq 1$.

**Proof.** Notice that $\delta_{b+1} = N_{\Delta k} r_{\bar{y}} + [\mathbf{0}\ I]\, \theta_{\Delta k}$. Then $\delta_{b+1}$ belongs to the Minkowski sum of $\{N_{\Delta k} r_{\bar{y}} : r_{\bar{y}}^T H_b^{-1} r_{\bar{y}}\} = \{r'_{\bar{y}} : r'^T_{\bar{y}} (N_{\Delta k}^T H_b N_{\Delta k})^{-1} r'_{\bar{y}} \leq 1\}$ and $\{[\mathbf{0}\ I]\, \theta_{\Delta k} : \theta_{\Delta k}^T \theta_{\Delta k} \leq \Theta_{\Delta k}^2\} = \{\theta'_{\Delta k} : \theta'^T_{\Delta k} H_\theta \theta'_{\Delta k} \leq 1\}$. Since both sets are ellipsoids, their Minkowski sum can be tightly outer-approximated by an ellipsoid. The rule in (24) is the outer-approximation that minimizes the square root of the sum of its principal axes (Kurzhanskiĭ and Vályi, 1997, Sec. 2.5).

*Corollary 10.* Without disturbances, the update law

$$H_{b+1} = N_{\Delta k}^T H_b N_{\Delta k} \quad (25)$$

satisfies $\delta_{b+1}^T H_{b+1}^{-1} \delta_{b+1} \leq 1$.

Algorithm 1 can then be used to select $\Delta k$, assigning the value of $\eta(\tilde{x}, H_b, \Delta k)$ to the variable $s$ at each iteration. The observer described in this section runs right after $\Delta k$ is computed to update $\bar{\psi}_0(k_{b+1})$ and $H_{b+1}$.

## 5. NUMERICAL EXAMPLES

We consider a linearized model of a batch reactor with a discrete-time output-feedback controller. The plant is the same as in (Donkers, 2011, Chap. 5, and references therein), with $n_p = 4, n_\psi = 2$, and $n_v = 2$. The controller of the form (2) is also the same, but running with $h = 0.01$:

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B_c = \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix},\ C_c = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix},\ D_c = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix},$$

We add a disturbance matrix $E^T = [1\ 0\ 0\ 0]$, set $\xi_p^T(0) = 10\,[1\ {-1}\ {-1}\ 1]$ and $\xi_c(0) = \mathbf{0}$. We then put the plant in the normal observable form. We set $\sigma = 0.1$, which satisfies Corollary 3 with $\rho = 0.6$ and $\gamma = 0.5$. The simulation is run both with $(W = 0.1)$ and without disturbance. When present, $\omega(t) = W$, if $t \leq 5$; $\omega(t) = 0$ otherwise. For the observer based STC of Sec. 4 we set $R = 1.5\|\bar{\psi}(0)\|$.

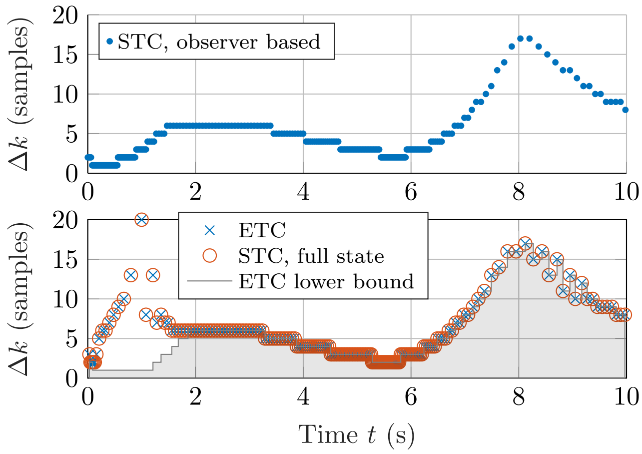


Fig. 1. Simulation results for STC and PETC without disturbance: STC triggering times, using the observer (top); ETC triggering times, STC triggering times with full state information and worst-case prediction using the STC logic with the observer (bottom).

On the unperturbed scenario, the triggering performances of the proposed STC strategies and the corresponding PETC are shown in Fig. 1. Notice that the triggering times for the STC start small and increase over time. This happens because the ellipsoids converge faster than the state norm in this example. Consequently, the observer based STC timing converges to the state based. To demonstrate that the STC logic can be used to predict a lower bound to the PETC triggering time, we added in Fig. 1 the estimated triggering time computed by the STC logic. The STC logic with full state information yields perfect prediction, as expected.

The results for the perturbed case are shown in Fig. 2. Clearly, control performances of both STC and PETC are very similar. Due to the disturbances, the STC triggering gets more conservative as the state norm approaches zero. As expected from Remark 5, after 5 time units the STC converges to periodic sampling – this is when the simulated disturbance vanishes. When the observer is used, the STC strategy gets more conservative in the first few time units of simulation. When used as a lower bound predictor, bounds are tight after 2 time units, when the observer ellipsoid has become sufficiently small. Later, as the state approaches the origin, the bounds converge to $\Delta k = 1$.

The observer performance during the first time unit is depicted in Fig. 3. We can see that the ellipsoid shrinks over time, while guaranteeing that $\bar{\psi}(k_b) - \bar{\psi}_0(k_b) \in \{r_{\bar{y}} : r_{\bar{y}}^T H_b^{-1} r_{\bar{y}} \leq 1\}$ for all time steps. Finally, Fig. 4 presents the regions $\mathcal{R}_{\Delta k}$ and $\mathcal{S}_{\Delta k}$ projected at the unmeasured subspace at the first triggering instant after 1.5 time units. We can see how the observer ellipsoid touches regions related to smaller triggering times, while the actual state is within a region related to a higher sampling time.

## 6. DISCUSSION AND FUTURE WORK

In this work we presented STC strategies for perturbed LTI systems under output feedback. We did so by devising algorithms that compute lower bounds to the corresponding PETC triggering times. This yields conservative STC

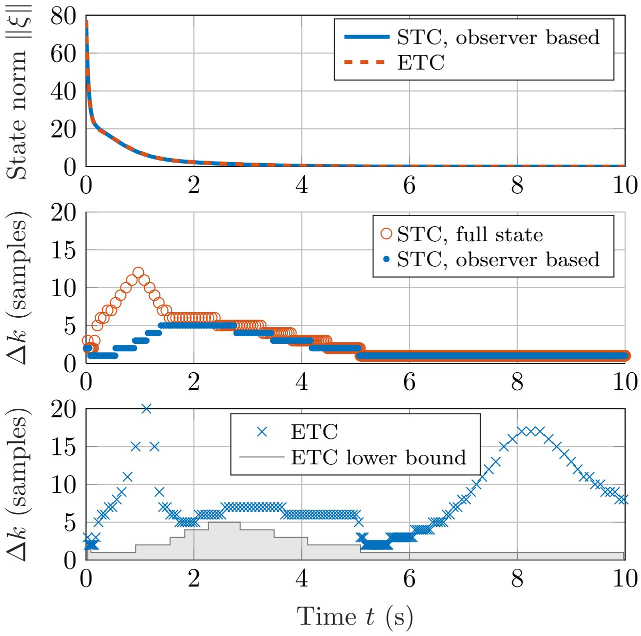


Fig. 2. Simulation results for the output-feedback STC and PETC, considering disturbances: state norm (top), STC triggering sample times (middle), and ETC triggering sample times with their corresponding worst-case prediction using the STC logic (bottom).

strategies that ensure stability and $\mathcal{L}_2$-gain performances. The STC algorithms are very simple for the case where the complete state is available. When not all states are measured, the STC logic is still relatively simple, but needs a special type of observer, which we developed. In the numerical example provided, after a certain time the output-based STC converges to the state-based STC (which yields the same sampling as the PETC), when disturbances are absent. Unfortunately, the combined prediction uncertainty due to absence of state information and presence of disturbances renders the STC rather conservative. We are working on improved observers with guaranteed ellipsoid convergence and reduced conservativeness.

Our algorithms can also be used as a building block to construct abstractions for ETC scheduling. The proposed state-space partitioning is promising in this sense as its cardinality does not scale with the number of states. We are currently working on such an extension. Other interesting extensions include: considering measurement noise; considering dynamic triggering functions; and extending the presented strategies to decentralized control.


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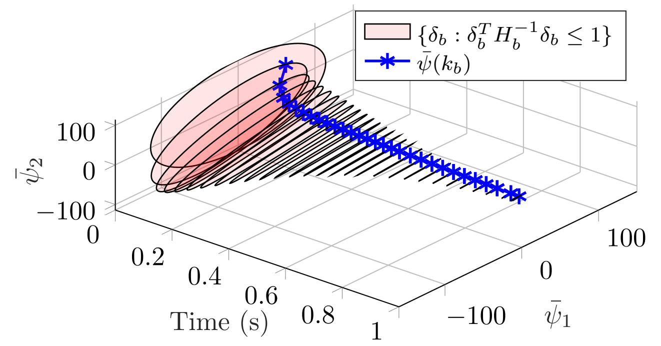


Fig. 3. Evolution of the observer ellipsoids and the actual unmeasured states $\bar{\psi}(k_b)$ for the perturbed scenario.

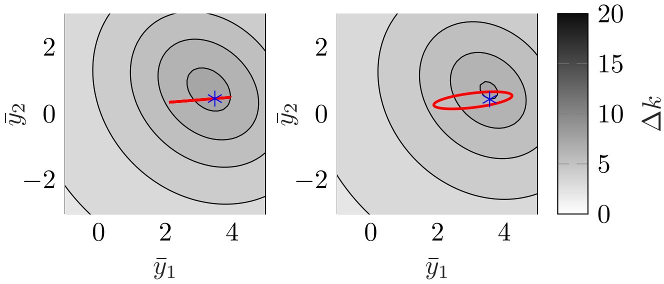


Fig. 4. $\mathcal{R}_{\Delta k}$ (right) and $\mathcal{S}_{\Delta k}$ (left) projected at $\bar{y}$ at a selected instant. The red ellipsoids show the observer bounds and the blue marks are the actual value of $\bar{y}$.


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